

Space Mission Design - AE 734

Orbit Requirement & Rocket Performance :-

$$F = \text{Thrust} - \boxed{Mg} - \boxed{D}$$

→ Drag by Air
→ Gravity by earth

$$\Rightarrow M \frac{dv}{dt} = \dot{m} V_e - Mg - D$$

$$\Rightarrow \frac{dv}{dt} = \frac{\dot{m}}{M} V_e - g - \frac{D}{M}$$

$$\Rightarrow \int_{v_i}^{v_f} \frac{dv}{dt} = \left\{ \frac{dM}{dt} \frac{V_e}{M} - g - \frac{D}{M} \right\} dt$$

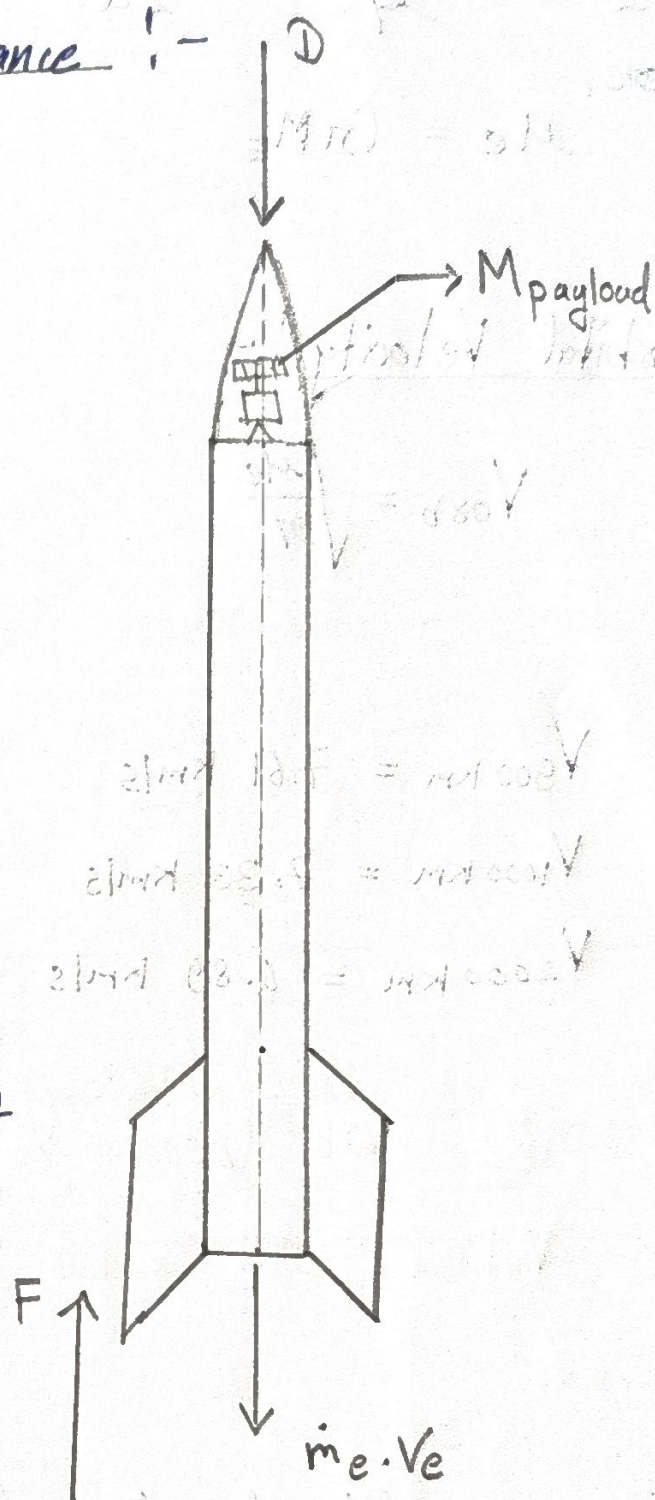
$$\text{So, } \Rightarrow v_f - v_i = V_e \int_{M_0}^{M_f} - \frac{dM}{M} - g(t_f - t_i) - \int \frac{D}{M} dt$$

$$\Rightarrow \Delta v = v_f - v_i = V_e \ln \frac{M_0}{M_f} - g(t_f - t_i) - \int_{t_i}^{t_f} \frac{D}{M} dt$$

either

① 430 m/s (At the equatorial orbit trajectory, rotational speed of earth)

② 0 m/s (for polar orbit, there will be no effect of earth rotation in the direction vertical to it)

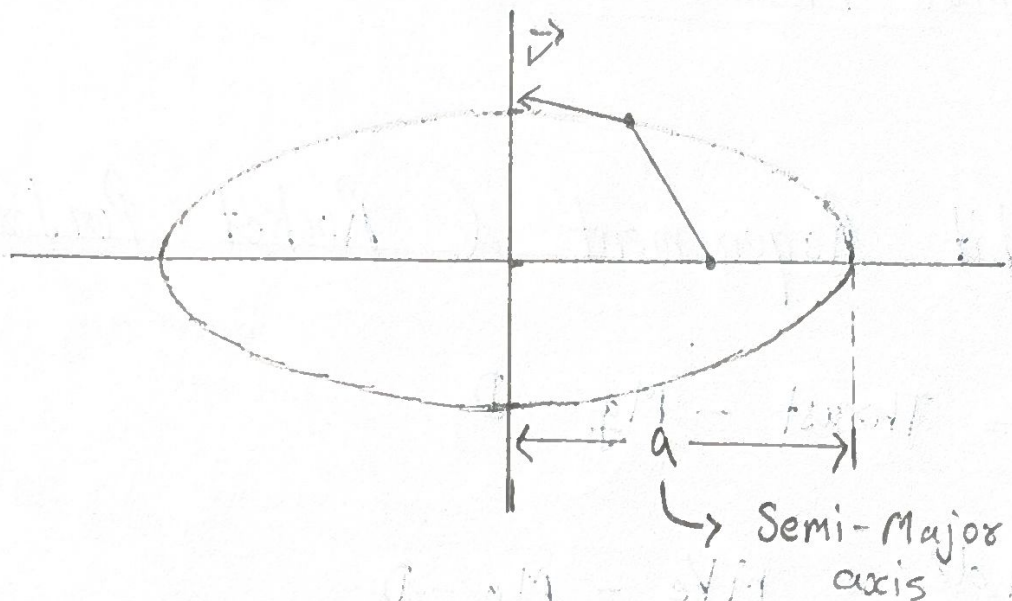


Vis-Viva Equation :-

$$\frac{V^2}{2} - \frac{\mu_e}{r} = -\frac{\mu_e}{2a}$$

Where,

$$\mu_e = GM_E$$



Orbital Velocity :-

$$V_{orb} = \sqrt{\frac{\mu_e}{r}}$$

So,

$$V_{500 \text{ km}} = 7.61 \text{ km/s}$$

$$V_{1000 \text{ km}} = 7.33 \text{ km/s}$$

$$V_{2000 \text{ km}} = 6.89 \text{ km/s}$$

But, these all are the orbiting Velocity Which is orbiting at the altitude of 500, 1000, 2000 kms, we need to give Total Velocity Such that first it should reach to that particular height then the orbital Velocity.

Basically, we need to give such kinetic energy such that the vehicle will reach to some height which has that potential energy.

T.E. $\rightarrow$ Stands for Total Energy

$$(T.E.)_{orbit} - (T.E.)_{earth \text{ surface}} = \frac{-\mu}{2a} - \left(\frac{-\mu}{R_E} \right)$$

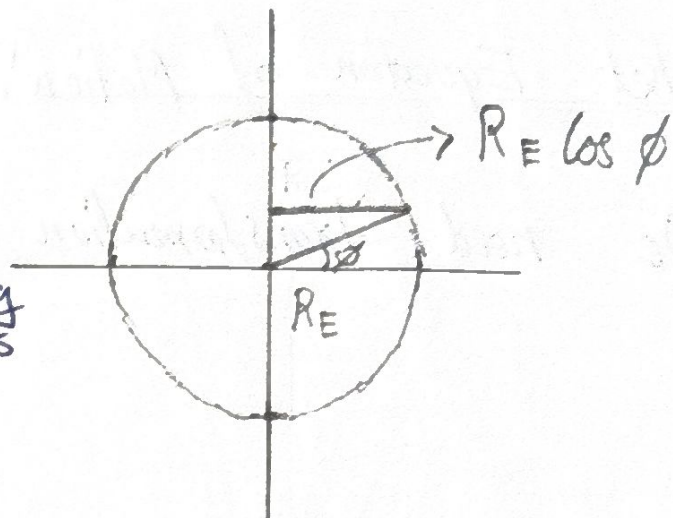
$$\therefore \frac{1}{2} V_{ideal}^2 = \frac{\mu}{R_E} - \frac{\mu}{2a}$$

practically, it will be $\frac{1}{2} V_{orbit}^2$, rest of the energy we are losing as the gravitational loss

$$\therefore V_{ideal} = \sqrt{\frac{2\mu}{R_E} - \frac{\mu}{a}}$$

$$\Rightarrow V_e \ln \left(\frac{M_o}{M_f} \right) = V_f - V_i + g_o(t_f - t_i) + \Delta V_{drag loss}$$

$$= W_E R_E \cos \phi \sin i$$



$$\Rightarrow \frac{V_a^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a} \quad \left. \vphantom{\frac{V_a^2}{2}} \right\} \text{Vis-Vira Equation}$$

The term $g_o(t_f - t_i)$ means — The required work done to the launch to be flown at some particular height stationarily.

Basically thrust should balance the weight.

$$V_e \ln \left(\frac{M_i}{M_f} \right) \rightarrow \sum_{k=1}^N V_e \ln \left(\frac{M_i}{M_f} \right)_k = \sqrt{\frac{2\mu}{R_E} - \frac{\mu}{a}} + \int_{t_0}^{t_s} g \sin \gamma dt + \int_{t_0}^t \frac{D}{M} dt + \Delta V_{steering loss} - \Delta V_{total}$$

8.18
(typically)

1.5 to 2
(typically)

0.3 to 0.5
(typically)

$$\Delta V = V_e \ln \left(\frac{M_i}{M_f} \right)$$

$$Isp \times g_o$$

$$450 \times 9.81$$

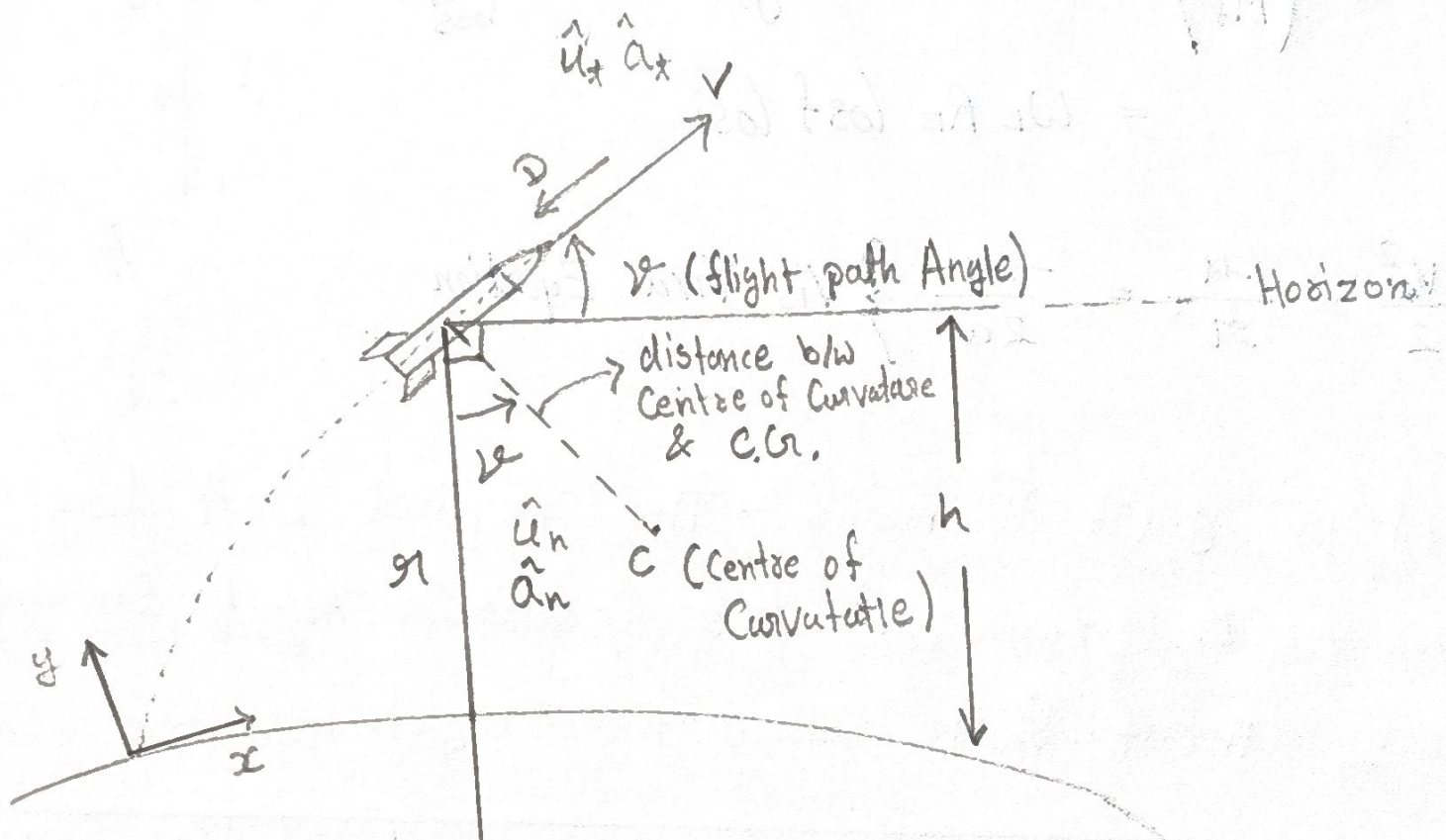
$$= V_e$$

1000 kg
(typically)

Date:- 09/01/2026

Rocket Equation of Motion:-

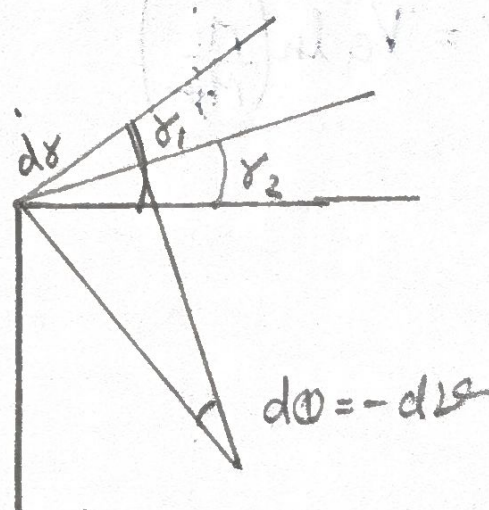
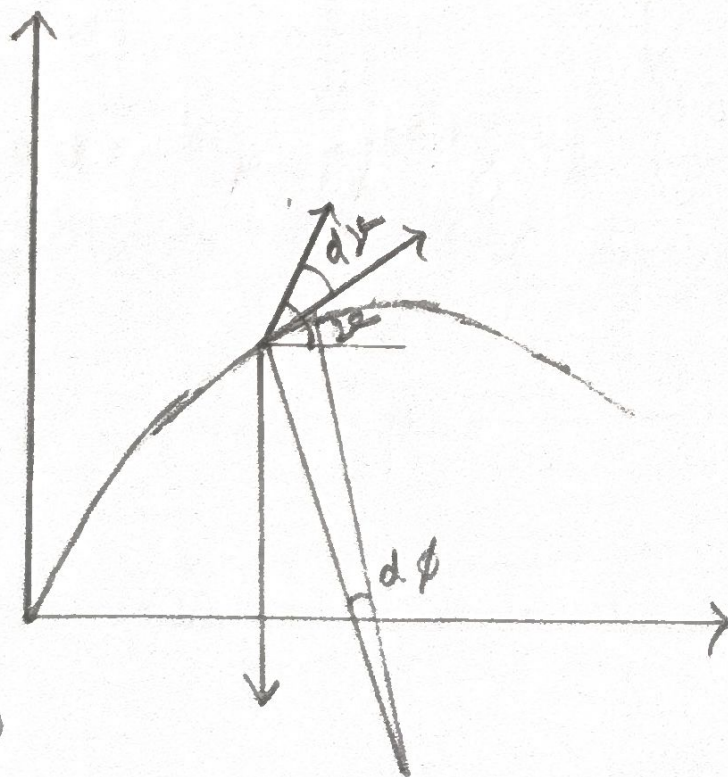
We need transformation frame body frame to Inertial frame.



mg
(Weight force towards the Centre of earth)

$$d\theta = -d\phi$$

$$d\sigma = \sigma d\phi$$



$$d\sigma = \sigma d\theta$$

$$\frac{d\sigma}{dt} = \sigma \frac{d\theta}{dt}$$

$$V = \sigma \left(-\frac{dr}{dt} \right) \Rightarrow \frac{V}{\sigma} = -\dot{r}$$

a_n due to Curvature,

$$a_n = \frac{v^2}{r} = v \cdot \frac{v}{r} = -v \frac{dv}{dt}$$

acceleration due to Curvature

$$a = \frac{v^2}{R \cos \theta}$$

So, that in the normal direction,

$$= \frac{v^2}{R \cos \theta} \cos \theta$$

$\Rightarrow \hat{u}_n$ direction,

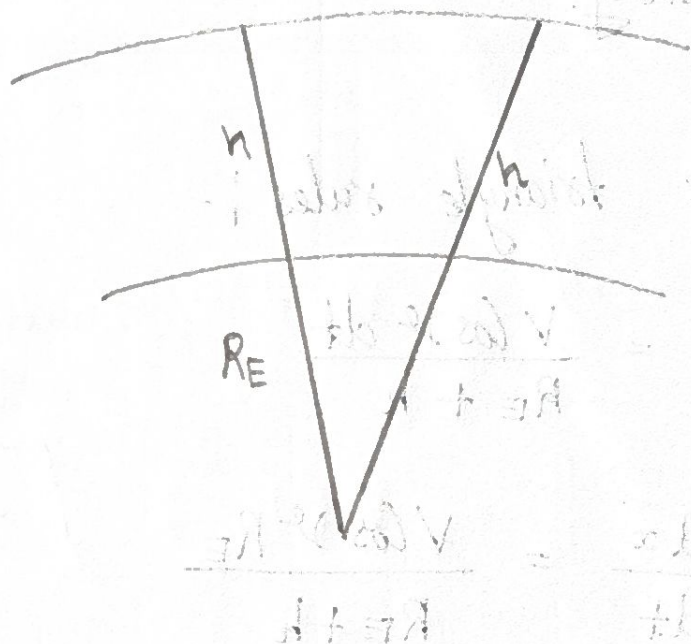
$$mg \cos \theta = m(a_n) \quad [\text{In the normal direction}]$$

$$\therefore mg \cos \theta = m \left[\frac{v^2}{R \cos \theta} \cos \theta - v \frac{dv}{dt} \right]$$

$$\boxed{v \frac{dv}{dt} = - \left(g - \frac{v^2}{R \cos \theta} \right) \cos \theta}$$

$\Rightarrow \hat{u}_t$ direction,

$$\boxed{T - D - mg \sin \theta = m \frac{dv}{dt}}$$



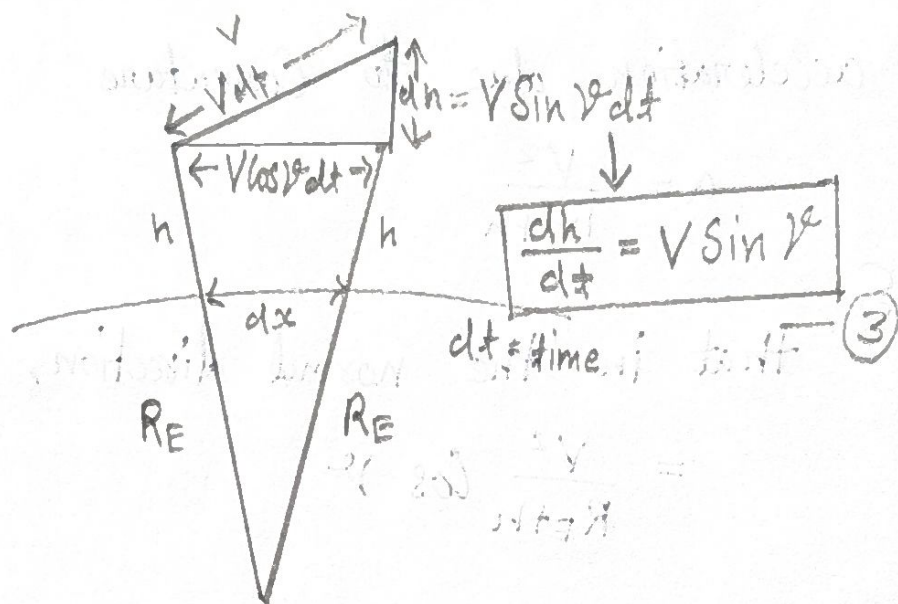
We need Coordinates of the Spacecraft, So, we will need additional information as x and h So that we can trace the trajectory.

from triangle rules:-

$$\frac{dx}{dE} = \frac{V \cos \varphi dt}{R_E + h}$$

$$\therefore \frac{dx}{dt} = \frac{V \cos \varphi R_E}{R_E + h}$$

$$\therefore \boxed{\frac{dx}{dt} = \frac{R_E V \cos \varphi}{R_E + h}} \quad (4)$$

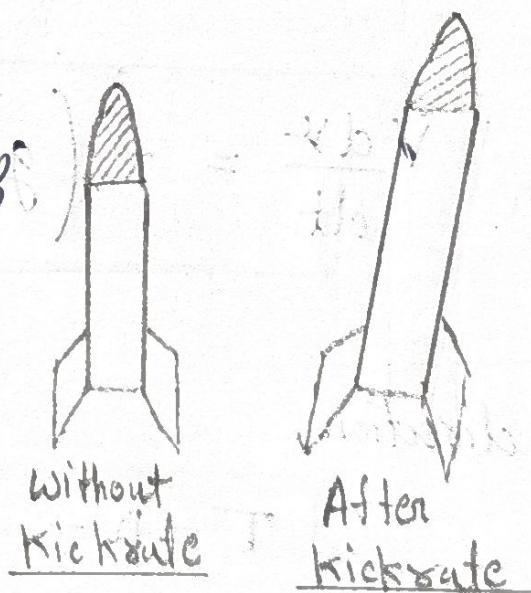


★ Gravity turn:- After reaching some height kick rate is given for turning that will change the orientation.

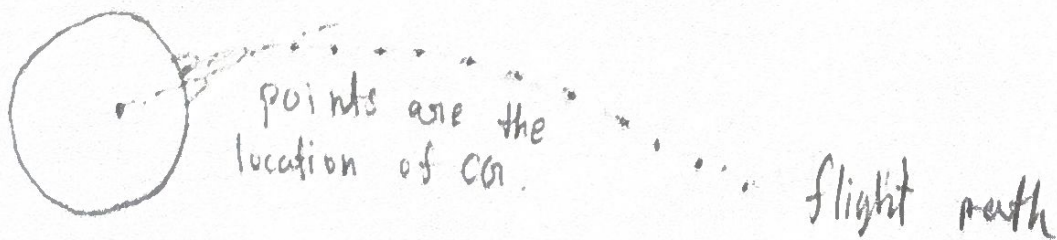
$$\frac{d\varphi}{dt} = -\frac{1}{V} \left(g - \frac{V^2}{R_E + h} \right) \cos \varphi \quad \left\{ \because \text{from eq. (3)} \right\}$$

So, that it will be the highest when the Velocity is lowest.

We need:- Angle of Attack has to be zero even we can the flight path Angle.



{It will Slightly be inclined and turn into the trajectory}



Date :- 13/01/2026

Launching Rocket

- Initial mass m_0
- Final mass m_f
- $\dot{m}_e = \text{Constant}$
- Neglect Drag, Variation of g with altitude (h)
- Final $h_{\max}$
- $\dot{m}_e$ for which $h \rightarrow h_{\max}$

Initial flight path Angle $\rightarrow 90^\circ$

Mass,

$$m(t) = m_0 - \dot{m}_e t$$

$$m_f = m_0 - \dot{m}_e t_{b_0}$$

$$t_{b_0} = \frac{m_0 - m_f}{\dot{m}_e}$$

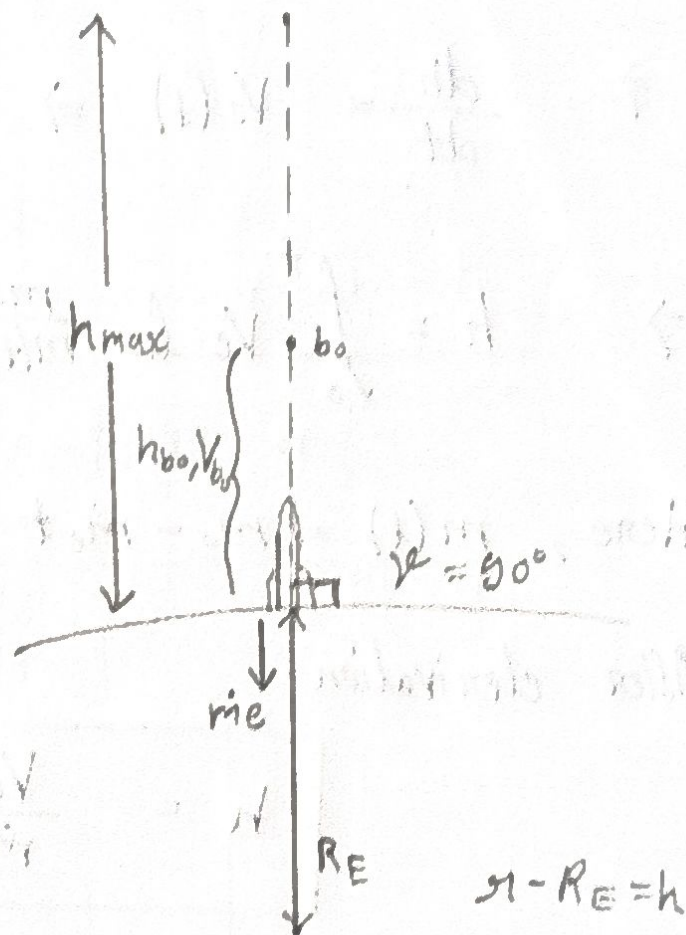
Force Equation,

$$M \frac{dv}{dt} = \dot{m}_e v_e - Mg$$

$$\frac{dv}{dt} = \frac{1}{M} \left(- \frac{dM}{dt} \right) v_e - g$$

On Integrating,

$$\int_{v_i}^{v_f} dv = - \int_{m_0}^{m_f} \frac{dm}{m} v_e - \int_{t_i}^{t_f} g dt$$



$\Rightarrow$

$$V(t) = V_e \ln \frac{m_0}{m(t)} - g_0 t_0$$

(1)

Now,

$$\Rightarrow \frac{dh}{dt} = V \sin \varphi, \text{ for constants } \varphi = 90^\circ$$

$$\Rightarrow \frac{dh}{dt} = V_e(t) \Rightarrow \int_0^h dh = \int_0^t V(t) dt$$

$$\Rightarrow h = \int_0^t V_e \ln \frac{m_0}{m(t)} - g_0(t)$$

Here, $m(t) = m_0 - \dot{m}_e t$

After derivation

$$h = \frac{V_e}{\dot{m}} \left[(m_0 - \dot{m}_e t) \ln \frac{m_0 - \dot{m}_e t}{m_0} + \dot{m}_e t \right] - \frac{g_0 t^2}{2}$$

(2)

Valid Only When thrust exists

to find h_{bo} , V_{bo} , put time $t = t_{bo} = \frac{m_0 - m_f}{\dot{m}_e}$

find this t_{bo} in (1), (2)

$$V_{bo} = V_e \ln \frac{m_0}{m_f} - g \frac{(m_0 - m_f)}{\dot{m}_e}$$

$$h_{bo} = \frac{V_e}{\dot{m}_e} \left[m_f \ln \frac{m_f}{m_0} + (m_0 - m_f) \right] - \frac{1}{2} \left(\frac{m_0 - m_f}{\dot{m}_e} \right)^2 g$$

After

$$V = u + at \rightarrow V = V_{b0} - g(t - t_{b0}) \quad \{t > t_{b0}\}$$

Now,

$$\frac{dh}{dt} = V \Rightarrow \int dh = \int V dt$$

$$\Rightarrow h - h_{b0} = V_{b0}(t - t_{b0}) - \frac{1}{2} g (t - t_{b0})^2$$

Substituting h_{b0}, V_{b0}, t_{b0} we get,

$$V = V_e \ln \frac{m_0}{m_f} - g_0(t) \quad \text{--- (3)} \quad (t > t_{b0})$$

$$h = \frac{V_e}{r_{ie}} \left[m_0 \ln \frac{m_f}{m_0} + (m_0 - m_f) \right] + V_e t \ln \frac{m_0}{m_f} - \frac{1}{2} g_0 t^2 \quad \text{--- (4)}$$

At $t_{max}, V=0$ Substitute in eq (3),

$$t_{max} = \frac{V_e}{g_0} \ln \frac{m_0}{m_f}$$

& from eq (4),

$$h_{max} = \frac{1}{2} \frac{V_e^2}{g_0} \ln^2 n - \frac{V_e m_0}{r_{ie}} \frac{(n \ln n) - (n-1)}{n} \quad \text{--- (5)}$$

Assumptions :- Vertical ascent, No drag, constant g_0 , constant thrust

Example :- find t_{b0} , h_{b0} , V_{b0} , h_{max}

$$M_0 = 68000 \text{ kg}$$

$$n = 15$$

$$I_{sp} = 390 \text{ s}$$

$$(T) \text{ Thrust} = 933.91 \text{ kN}$$

$$\Rightarrow V_e = I_{sp} \times g_0 = 390 \times 9.81 = 3825.9 \text{ m/s}$$

$$\Rightarrow T = \dot{m}_e V_e$$

$$\Rightarrow \dot{m}_e = \frac{T}{V_e} = \frac{933.91 \text{ kN}}{3825.9 \text{ m/s}} = 244.1 \text{ kg/s}$$

$$\Rightarrow n = 15 = \frac{M_0}{m_f} \Rightarrow m_f = \frac{M_0}{15} = \frac{68000}{15} = 4533.33 \text{ kg}$$

$$\Rightarrow t_{b0} = \frac{M_0 - m_f}{\dot{m}_e} = 260 \text{ s}$$

$$\Rightarrow V_{b0} = V_e \ln \frac{M_0}{m_f} - g_0 t_{b0} = 3825.9 \times \ln \left(\frac{68000}{4533.33} \right) - 9.81 \times 260$$

$$\Rightarrow V_{b0} = 7.81 \text{ km/s}$$

$$\Rightarrow h_{b0} = \frac{V_e}{\dot{m}_e} m_f \ln \left(\frac{m_f}{M_0} \right) + V_e t_{b0} - \frac{1}{2} g_0 t_{b0}^2$$

$$\Rightarrow h_{b0} = \frac{3825.9}{244.1} \times 4533.33 \ln \left(\frac{1}{15} \right) + 3825.9 \times 260 - \frac{1}{2} \times 9.81 \times 260^2$$

$$\Rightarrow h_{b0} = 470.74 \text{ km}$$

$$\Rightarrow h_{\max} = \frac{1}{2} \frac{V_e^2}{g} \ln^2 n - \frac{V_e m_0}{m_e} \frac{(n \ln n) - (n-1)}{n}$$

$$\Rightarrow h_{\max} = \frac{1}{2} \frac{3825.9^2}{9.81} \ln^2 15 - \frac{3825.9 \times 68000}{244.1} \left(\frac{15 \ln 15 - 14}{15} \right)$$

$$\Rightarrow h_{\max} = 3578.9 \text{ km}$$

Restarted Staging in field free Space

Date: - 14/01/2026

Restricted Staging: - Some conditions are already fixed i.e. Same exhaust Velocity for all stages, Same thrust to weight ratio, Same burn time Δt for each stage, Same structural mass fraction.

field free Space: - Independent of gravity (No gravity)

$$m_e = \frac{T}{V_e} = - \frac{dm}{dt}$$

$$dm = \frac{-T}{I_{sp} \times g_0} dt \quad \left\{ \text{Consider constant } T, I_{sp} g_0 \right\}$$

$$\Rightarrow \Delta t = \frac{I_{sp} \times g_0 (m_0 - m_f)}{T}$$

$$\Rightarrow \Delta t = \frac{I_{sp} \times g_0}{T} m_0 \left(1 - \frac{m_f}{m_0} \right)$$

$$\frac{m_0}{m_f} = n \Rightarrow \text{mass ratio}$$

$$\Rightarrow \Delta t = \frac{I_{sp} g_0}{T} m_0 \left(1 - \frac{1}{n} \right)$$

$$\Rightarrow \Delta t = \frac{I_{sp}}{T/g_0 m_0} \left(\frac{n-1}{n} \right)$$

$$\Rightarrow \Delta t = \frac{I_{sp}}{T/w} \left(\frac{n-1}{n} \right)$$

$$\Rightarrow \Delta V = I_{sp} \times g_0 \ln \left(\frac{m_0}{m_f} \right)$$

$$\Rightarrow \frac{m_0}{m_f} = e^{\frac{\Delta V}{I_{sp} \times g_0}}$$

$$\Rightarrow \frac{m_0}{m_0 - \Delta m} = e^{\frac{\Delta V}{I_{sp} \times g_0}}$$

$$\Rightarrow \frac{m_0 - \Delta m}{m_0} = \frac{1}{e^{\frac{\Delta V}{I_{sp} \times g_0}}}$$

$$\Rightarrow \boxed{\frac{\Delta m}{m_0} = 1 - \frac{1}{e^{\frac{\Delta V}{I_{sp} \times g_0}}}}$$

Thrust to weight ratio (T/w)

↳ PSLV = 1.2 to 1.5

Geosynchronous Rocket = 5

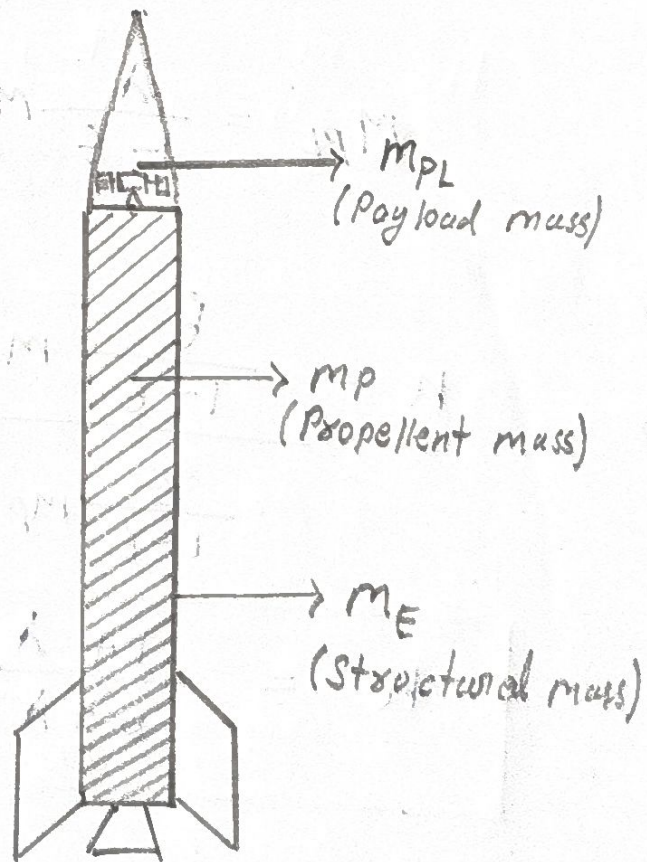
$$\Rightarrow m_0 = m_E + m_p + m_{PL}$$

$$\Rightarrow \frac{m_0}{m_0} = \frac{m_E}{m_0} + \frac{m_p}{m_0} + \frac{m_{PL}}{m_0}$$

$$\Rightarrow 1 = \underbrace{\pi_E}_{\text{Structural fraction}} + \underbrace{\pi_p}_{\text{propellant fraction}} + \underbrace{\pi_{PL}}_{\text{payload fraction}}$$

So,

$$\Rightarrow n = \frac{m_0}{m_f} = \frac{m_E + m_p + m_{PL}}{m_E + m_{PL}}$$



Single Stage Vehicle

$$\Rightarrow \Delta V = I_{sp} \times g_0 \ln(n) \quad \rightarrow \text{we need } n \text{ as the function of } (\lambda, \epsilon) \text{ i.e. } n = f(\lambda, \epsilon)$$

$$\Rightarrow \text{payload ratio} = \frac{m_{PL}}{m_E + m_P} = \lambda$$

$$\Rightarrow \text{Structured ratio} = \frac{m_E}{m_E + m_P} = \epsilon$$

$$\Rightarrow n = \frac{m_E + m_P + m_{PL}}{m_E + m_{PL}}$$

$$\Rightarrow \epsilon = \frac{m_E}{m_E + m_P}$$

$$\Rightarrow m_E = \frac{\epsilon}{1-\epsilon} m_P$$

$$\Rightarrow m_{PL} = \lambda (m_E + m_P)$$

$$\Rightarrow m_{PL} = \frac{\lambda}{1-\epsilon} m_P$$

$$\Rightarrow n = \frac{\frac{\epsilon}{1-\epsilon} m_P + m_P + \frac{\lambda}{1-\epsilon} m_P}{\frac{\epsilon}{1-\epsilon} m_P + \frac{\lambda}{1-\epsilon} m_P} =$$

$$\Rightarrow \boxed{n = \frac{1+\lambda}{\epsilon+\lambda}}$$

S₀,

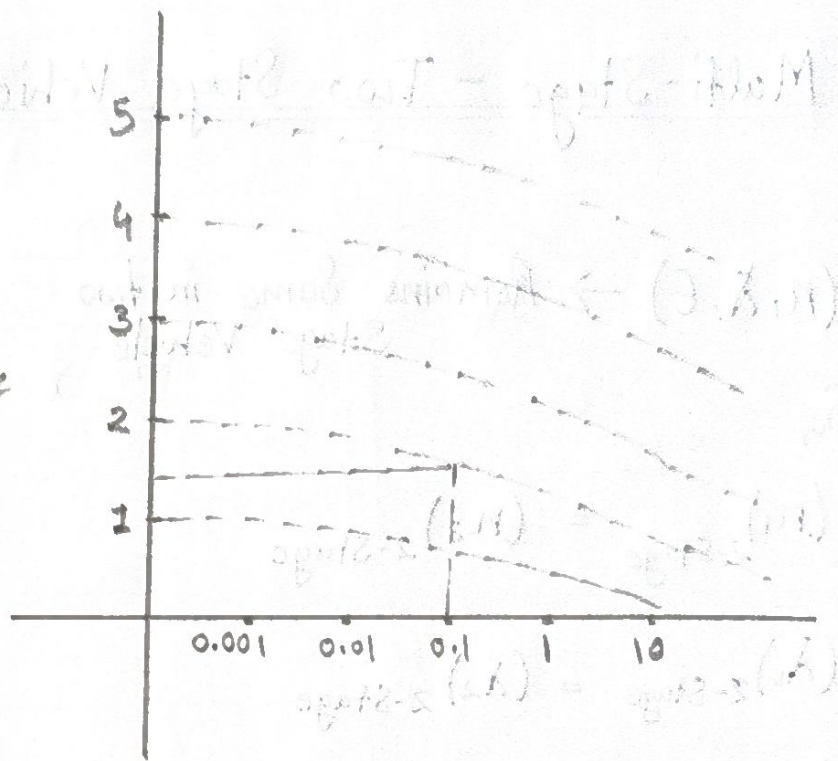
$$\Delta v = I_{sp} \times g_0 \ln \left(\frac{1 + \lambda}{\epsilon + \lambda} \right)$$

We can also plot a Rocket performance graph from this equation.

$$\frac{\Delta v}{I_{sp} \times g_0} = \ln \left(\frac{1 + \lambda}{\epsilon + \lambda} \right)$$

$$\frac{\Delta v}{I_{sp} \times g_0} = 1.94 \quad \left\{ \text{Typically} \right\}$$

$$\lambda = 0.01$$



Multi-Stage - Two Stage Vehicle

Date: 15/01/2026

$(n, \lambda, \epsilon) \rightarrow$ Remains Same in two Stage Vehicle

So,

$$(n_1)_{2\text{-stage}} = (n_2)_{2\text{-stage}}$$

$$(\lambda_1)_{2\text{-stage}} = (\lambda_2)_{2\text{-stage}}$$

$$(\epsilon_1)_{2\text{-stage}} = (\epsilon_2)_{2\text{-stage}}$$

$$(n)_{\text{single stage}} \neq (n_1) \text{ or } (n_2) \text{ of two-stage}$$

Just to remember and compare,

In Single Stage,

$$\Rightarrow n = \frac{m_0}{m_f} = \frac{M_E + M_P + M_{PL}}{M_E + M_{PL}} = \frac{1 + \lambda}{\epsilon + \lambda}$$

Restricted Staging in field free Space

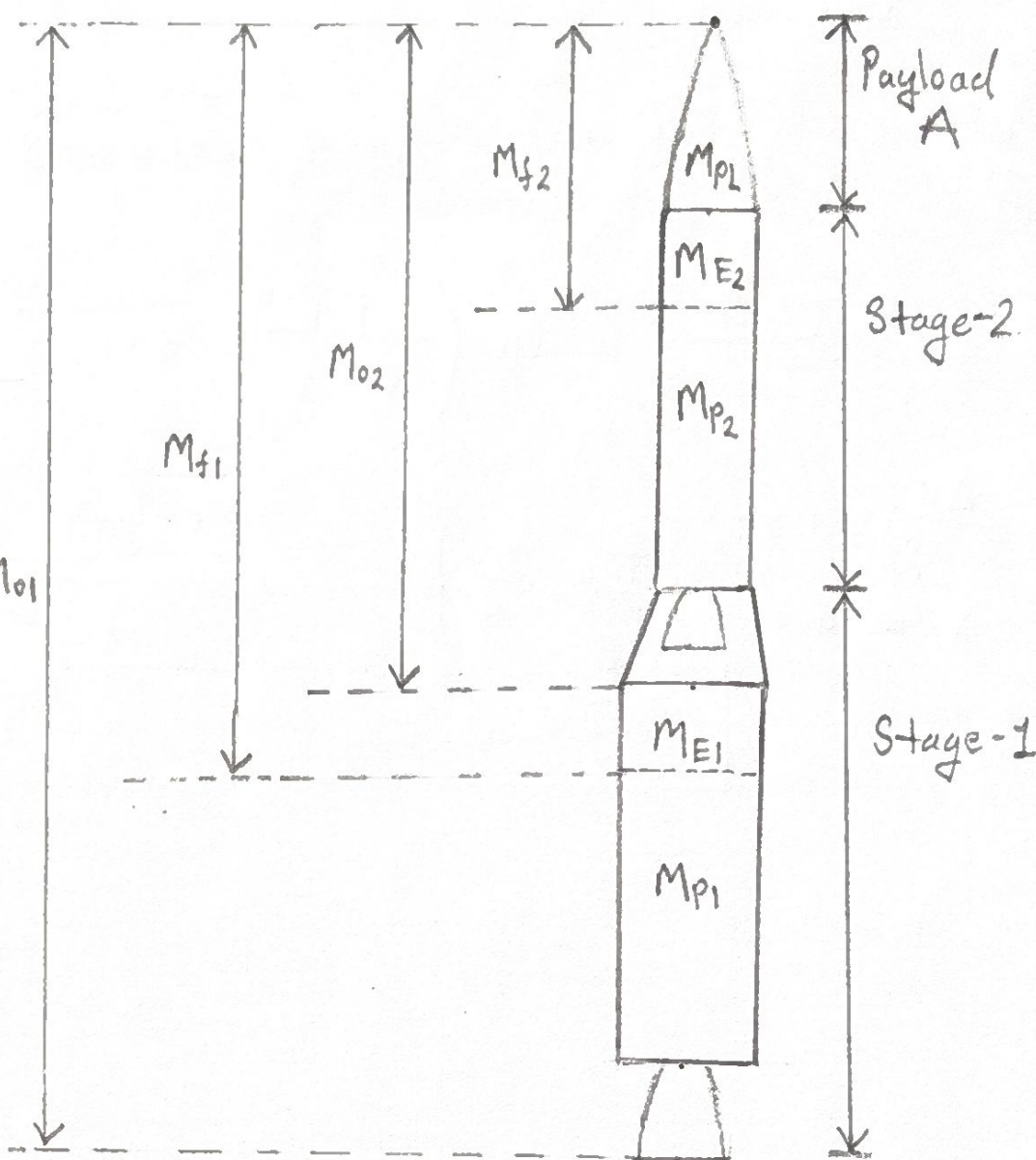
Single Stage :-

$$\Delta V = I_{sp} \cdot g_0 \ln(n) \quad \text{--- (1)}$$

We need n as a function of π_{PL} & ϵ i.e. $n = f(\pi_{PL}, \epsilon)$

$$\pi_{PL} = \frac{M_{PL}}{M_0}$$

$$\lambda = \frac{M_{PL}}{M_E + M_P} = \frac{M_{PL}}{M_0 - M_{PL}} = \frac{M_{PL}}{M_{PL}} \left(\frac{1}{\frac{M_0}{M_{PL}} - 1} \right)$$



$$\Rightarrow \lambda = \frac{1}{\frac{1}{\pi_{PL}} - 1} = \frac{\pi_{PL}}{1 - \pi_{PL}}$$

$$\Rightarrow h = \frac{1 + \lambda}{\epsilon + \lambda} = \frac{1 + \frac{\pi_{PL}}{1 - \pi_{PL}}}{\epsilon + \frac{\pi_{PL}}{1 - \pi_{PL}}}$$

$$\Rightarrow h = \frac{1}{\epsilon + \pi_{PL}(1 - \epsilon)}$$

So, eq ① becomes,

$$\Rightarrow \Delta V = I_{sp} g_0 \ln \left(\frac{1}{\epsilon + \pi_{PL}(1 - \epsilon)} \right)$$

Now, we will derive a result about ϵ_a & ϵ_b of Single Stage & two Stage respectively.

Single Stage

$$\Rightarrow \epsilon_a = \frac{M_{\epsilon_{total}}}{M_{\epsilon_{total}} + M_{p_{total}}}$$

$$\Rightarrow \epsilon_a = \frac{M_{\epsilon_1} + M_{\epsilon_2}}{M_{\epsilon_1} + M_{\epsilon_2} + M_{p_1} + M_{p_2}}$$

Two Stage

$$M_{\epsilon_1} = \frac{\epsilon_2}{(1 - \epsilon_2)} M_{p_1}$$

$$M_{p_1} = \frac{(1 - \epsilon_b)}{\epsilon_b} M_{\epsilon_1}$$

$$M_{p_2} = \frac{(1 - \epsilon_b)}{\epsilon_b} M_{\epsilon_2}$$

By Substituting these Values,

we get,

$$\Rightarrow \epsilon_a = \epsilon_b$$

Now, we need n as a function of π_{PL} & ϵ i.e. $n = f(\pi_{PL}, \epsilon)$ in two stage compression:

S_{01}

$\Rightarrow$

$$\lambda_1 = \lambda_2$$

$\Rightarrow$

$$\frac{M_{PL}}{M_{E1} + M_{PL}} = \frac{M_{02}}{M_{01} - M_{02}} = \frac{M_{PL}}{M_{02} - M_{PL}} = \lambda_1$$

&

$\Rightarrow$

$$\lambda_2 = \frac{M_{PL}}{M_{E2} + M_{PL}} = \frac{M_{PL}}{M_{02} - M_{PL}}$$

S_{01}

$\Rightarrow$

$$\frac{M_{PL}}{M_{02} - M_{PL}} = \frac{M_{02}}{M_{01} - M_{02}}$$

$\Rightarrow$

$$M_{PL}(M_{01} - M_{02}) = M_{02}(M_{02} - M_{PL})$$

$\Rightarrow$

$$M_{02} = \sqrt{M_0} \sqrt{M_{PL}}$$

&

$\Rightarrow$

$$\pi_{PL} = \frac{M_{PL}}{M_0} \Rightarrow M_0 = \frac{M_{PL}}{\pi_{PL}}$$

S_{01}

$\Rightarrow$

$$M_{02} = \sqrt{\frac{M_{PL}}{\pi_{PL}}} \sqrt{M_{PL}}$$

$\Rightarrow$

$$M_{02} = \sqrt{\frac{1}{\pi_{PL}}} M_{PL}$$

S_{01}

$\Rightarrow$

$$\lambda_2 = \frac{M_{PL}}{M_{02} - M_{PL}} = \frac{M_{PL}}{\sqrt{\frac{1}{\pi_{PL}}} M_{PL} - M_{PL}} = \frac{1}{\sqrt{\frac{1}{\pi_{PL}}} - 1}$$

S_{01}

$\Rightarrow$

$$\lambda_2 = \frac{\pi_{PL}^{y_2}}{1 - \pi_{PL}^{y_2}}$$

Then,

$\Rightarrow$

$$n_2 = \frac{1 + \lambda}{\epsilon + \lambda} = \frac{1 + \frac{\pi_{PL}^{y_2}}{1 - \pi_{PL}^{y_2}}}{\epsilon + \frac{\pi_{PL}^{y_2}}{1 - \pi_{PL}^{y_2}}} = \frac{1}{\pi_{PL}^{y_2} (1 - \epsilon) + \epsilon}$$

$\Rightarrow$

$$\Delta V = \Delta V_1 + \Delta V_2$$

$\Rightarrow$

$$\Delta V = I_{sp} \times g_0 \ln(n_1) + I_{sp} \times g_0 \ln(n_2)$$

$\Rightarrow$

$$\Delta V = 2 \times I_{sp} \times g_0 \ln(n_2) \quad \{n_1 = n_2\}$$

$\Rightarrow$

$$\Delta V = 2 \times I_{sp} \times g_0 \ln\left(\frac{1}{\pi_{PL}^{y_2} (1 - \epsilon) + \epsilon}\right)$$

$\Rightarrow$

$$\Delta V = I_{sp} \times g_0 \ln\left(\frac{1}{\pi_{PL}^{y_2} (1 - \epsilon) + \epsilon}\right)^2$$

Now, we will transform $M_{\epsilon 1}$, $M_{\epsilon 2}$, M_{PL} & M_P in the form of π_{PL} , ϵ & M_{PL} .

$\Rightarrow$

$$\epsilon = \frac{M_{\epsilon 1}}{M_{01} - M_{02}}$$

$\Rightarrow$

$$M_{\epsilon 1} = \frac{(1 - \pi_{PL}^{y_2})}{\pi_{PL}} \epsilon M_{PL}$$

$\Rightarrow$

$$M_{\epsilon 2} = \frac{(1 - \pi_{PL}^{y_2})}{\pi_{PL}^{y_2}} \epsilon M_{PL}$$

$$\Rightarrow M_{p1} = \left[\frac{(1 - \pi_{PL}^{\gamma_2})(1 - \epsilon)}{\pi_{PL}} \right] M_{PL}$$

$$\Rightarrow M_{p2} = \left[\frac{(1 - \pi_{PL}^{\gamma_2})(1 - \epsilon)}{\pi_{PL}^{\gamma_2}} \right] M_{PL}$$

N-Stage "

$$\Rightarrow \Delta V = I_{sp} \times g_0 \times N \ln \left(\frac{1}{\pi_{PL}^{\gamma_N}(1 - \epsilon) + \epsilon} \right)$$

for large N,

$$\Rightarrow \pi_{PL}^{\gamma_N} \rightarrow 1 + \frac{1}{N} \ln(\pi_{PL}) \quad \left\{ \begin{array}{l} \text{Taylor Series} \\ \text{Expansion} \end{array} \right\}$$

$$\Rightarrow \Delta V = I_{sp} \times g_0 \times N \ln \left(\frac{1}{(1 + \frac{1}{N} \ln(\pi_{PL}))(1 - \epsilon) + \epsilon} \right)$$

As $N \rightarrow \infty$

$$\Rightarrow \Delta V = I_{sp} \times g_0 \times (1 - \epsilon) \ln \left(\frac{1}{\pi_{PL}} \right)$$

Calculate ΔV :-

$$\pi_{PL} = 0.05$$

$$\epsilon = 0.15$$

$$\Delta V = 7.$$

$$I_{sp} = 350 \text{ s}$$

for 3-Stage Vehicle,

$$\Rightarrow \Delta V = I_{sp} \times g_0 \times N \ln \left[\frac{1}{\left(1 + \frac{1}{N} \ln \pi_{PL}\right) (1-\epsilon) + \epsilon} \right]$$

$$\Rightarrow \Delta V = 7.928 \text{ km/s}$$

for N-Stage Vehicle ($N \rightarrow \infty$),

$$\Rightarrow \Delta V = I_{sp} \times g_0 \times (1-\epsilon) \ln \left(\frac{1}{\pi_{PL}} \right)$$

$$\Rightarrow \Delta V = 8.742 \text{ km/s}$$

Optimal Staging

Date:- 20/02/2026

$\Rightarrow f(x, y) \rightarrow$ function

$\Rightarrow g(x, y) = 0 \rightarrow$ constraint

find min/max of $f(x, y)$ Subject to $g(x, y) = 0$

$$\Rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

Generate,

$$\Rightarrow h(x, y, \eta)$$

$$\Rightarrow h(x, y, \eta) = f(x, y) + \eta g(x, y)$$

$$\Rightarrow dh = 0$$

$$\Rightarrow \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial \eta} d\eta = 0$$

$$\Rightarrow df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$$

$$\Rightarrow g(x, y) = 0, dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\partial g / \partial x}{\partial g / \partial y}$$

$$\therefore \frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y} = \frac{-\partial g / \partial x}{\partial g / \partial y}$$

$$\therefore \frac{\partial f / \partial x}{\partial g / \partial x} = \frac{\partial f / \partial y}{\partial g / \partial y} = -\eta$$

$\Rightarrow$

$$\therefore \frac{\partial f}{\partial x} + \eta \frac{\partial g}{\partial x} = 0$$

 $\Rightarrow$

$$\frac{\partial f}{\partial y} + \eta \frac{\partial g}{\partial y} = 0$$

If we combine both f & g then function h will get like

 $\Rightarrow$

$$h(x, y, \eta) = f(x, y) + \eta g(x, y)$$

 $\Rightarrow$

$$\text{for } dh = 0$$

 $\Rightarrow$

$$\therefore \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy + \frac{\partial h}{\partial \eta} d\eta = 0$$

 $\Rightarrow$

$$\therefore \left(\frac{\partial f}{\partial x} + \eta \frac{\partial g}{\partial x} \right) dx + \left(\frac{\partial f}{\partial y} + \eta \frac{\partial g}{\partial y} \right) dy + g(x, y) d\eta = 0$$

 $\Rightarrow$

$$d^2 h = \frac{\partial^2 h}{\partial x^2} dx^2 + 2 \frac{\partial^2 h}{\partial x \partial y} dx dy + \frac{\partial^2 h}{\partial y^2} dy^2 = -m \rightarrow \text{min} \\ +m \rightarrow \text{max}$$

 $\Rightarrow$

$$(i) \quad z = f(x, y) = -x^2 - y^2 \rightarrow \text{no constant}$$

 $\Rightarrow$

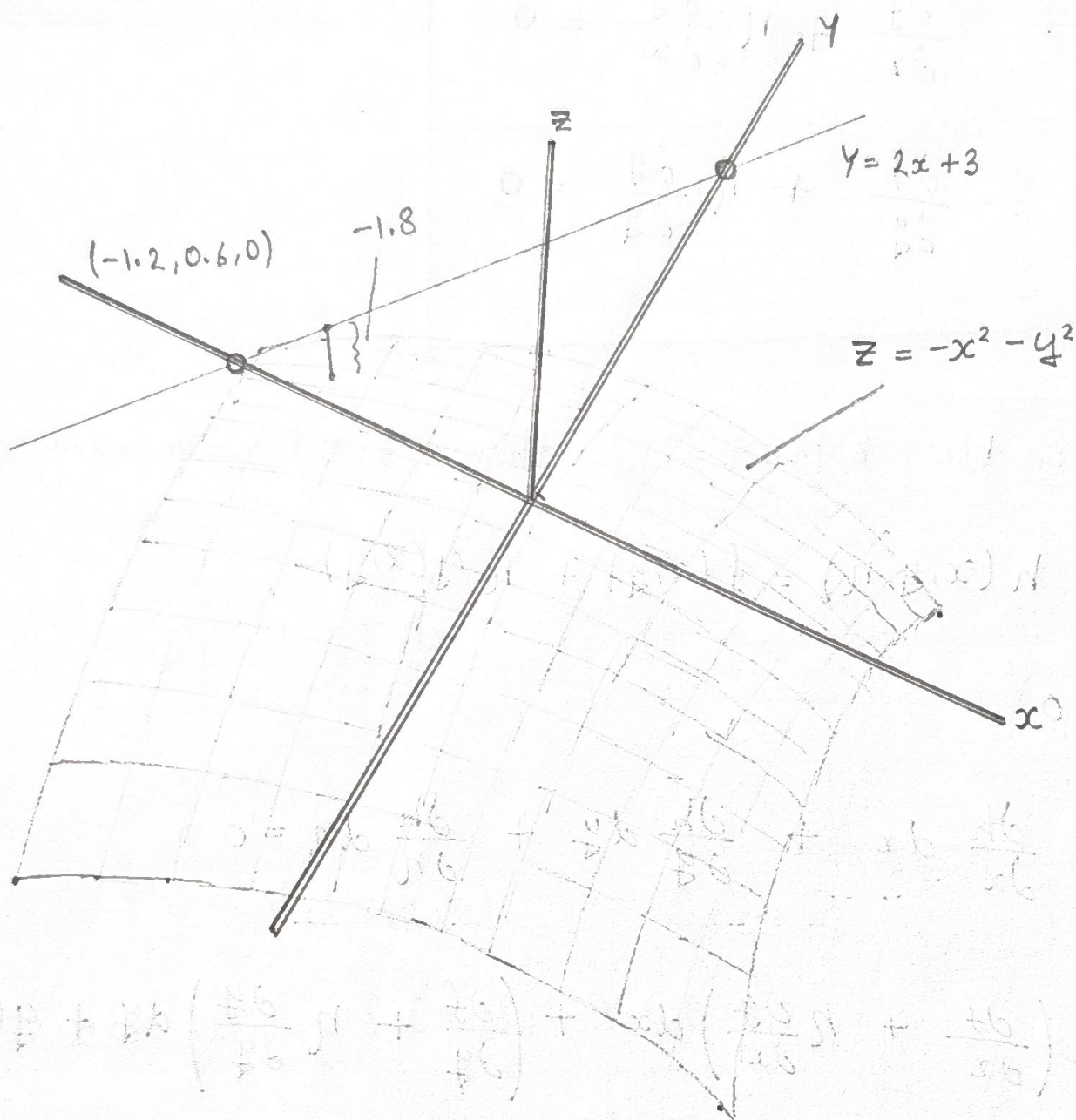
$$(ii) \quad z = f(x, y) = -x^2 - y^2 \rightarrow g(x, y) = y - 2x - 3 = 0 \rightarrow y = 2x + 3$$

 $\Rightarrow$

$$h(x, y, \eta) = f(x, y) + \eta g(x, y)$$

 $\Rightarrow$

$$h(x, y, \eta) = (-x^2 - y^2) + \eta(y - 2x - 3)$$



$$\Rightarrow dh = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \eta \frac{\partial g}{\partial x} = 0 \quad \& \quad \text{here } f = -x^2 - y^2 \& \ g = y - 2x - 3$$

$$\Rightarrow \frac{\partial f}{\partial y} + \eta \frac{\partial g}{\partial y} = 0$$

$$\Rightarrow \begin{aligned} \therefore -2x + \eta(-2) &= 0 \Rightarrow \boxed{x = -\eta} = -1.2 \\ \therefore -2y + \eta(1) &= 0 \Rightarrow \boxed{y = \eta/2} = 0.6 \end{aligned}$$

Because, $y - 2x - 3 = 0$

$$\therefore \eta = \frac{6}{5} = 1.2$$

Consider two Stage in field free Space :- (not restricted body)
no gravity

V_{e1}, V_{e2}, η_1	$\Delta V_{\text{increase}}$ or V_{bo} (as in book)
$\epsilon_1, \epsilon_2, \eta_2$	

$\Rightarrow$ Either we can increase $\pi_{PL} = \frac{m_{PL}}{m_0}$

$\Rightarrow$ or decrease $\frac{m_0}{m_{PL}} = \frac{1}{\pi_{PL}}$

$\Rightarrow$ $\therefore f(x, y) = \frac{m_0}{m_{PL}} \longrightarrow f(\eta_1, \eta_2)$

$\downarrow$
 η_1, η_2

$\Rightarrow$ Step mass m :-

$\Rightarrow$ $m_i = m_{Ei} + m_{Pi}$

$\Rightarrow$ $\therefore \frac{m_{Ei}}{m_{Ei} + m_{Pi}} = \epsilon_i$

$\Rightarrow$ total mass of rocket excluding payload :-

$\Rightarrow$ $M = \sum_{i=1}^{N=2} m_i$

$\Rightarrow$ $m_0 = M + m_{PL}$

$\Rightarrow$ Two Stage rocket :-

$m_0 = m_1 + m_2 + m_{PL}$

$\Rightarrow$ $\therefore \frac{1}{\pi_{PL}} = \frac{m_0}{m_{PL}} \Rightarrow \frac{m_1 + m_2 + m_{PL}}{m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{m_1 + m_{PL}}{m_{PL}}$

Two Stage : Continued :-

Date :- 21/01/2026

$$\Rightarrow \frac{1}{\pi_{PL}} = \frac{m_0}{m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} \cdot \frac{m_2 + m_{PL}}{m_{PL}} \rightarrow f(n_1, n_2)$$

{ We want to minimize the function and express it as a }
{ function of n_1 & n_2 }

Let,

$$\Rightarrow n_1 = \frac{m_1 + m_2 + m_{PL}}{m_{E1} + m_2 + m_{PL}} = \frac{m_1 + m_2 + m_{PL}}{\epsilon_1 m_1 + m_2 + m_{PL}}$$

Similarly

$$\Rightarrow n_2 = \frac{m_2 + m_{PL}}{m_{E2} + m_{PL}} = \frac{m_2 + m_{PL}}{\epsilon_2 m_2 + m_{PL}}$$

$$\Rightarrow \therefore n_2 (\epsilon_2 m_2 + m_{PL}) = m_2 + m_{PL}$$

$$\Rightarrow \therefore n_2 \epsilon_2 m_2 + n_2 m_{PL} - m_2 - m_{PL} = 0$$

$$\Rightarrow \therefore m_2 = \frac{(1 - n_2) m_{PL}}{n_2 \epsilon_2 - 1}$$

$$\Rightarrow m_2 = \frac{(n_2 - 1)}{(1 - n_2 \epsilon_2)} \cdot m_{PL} \rightarrow \text{Very important}$$

Similarly,

$$\Rightarrow m_1 = \frac{(n_1 - 1)}{(1 - n_1 \epsilon_1)} \cdot (m_2 + m_{PL})$$

⇒

$$\frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL}} = \frac{1 - \epsilon_1}{1 - \epsilon_1} \cdot \frac{m_1 + m_2 + m_{PL}}{m_2 + m_{PL} + \epsilon_1 m_1 + \epsilon_2 m_2} \cdot \frac{1}{\frac{\epsilon_1 m_1 + m_2 + m_{PL}}{1}}$$

$$\frac{(1 - \epsilon_1) n_1}{1 - \epsilon_1 n_2}$$

{ multiply, expand, neglect higher order terms }

$$\epsilon_1 m_1 + m_2 + m_{PL} - \epsilon_1 (m_1 + m_2 + m_{PL})$$

n_1

$$\frac{m_1 + m_2 + m_{PL}}{\epsilon_1 m_1 + m_2 + m_{PL}}$$

⇒

$$\frac{(1 - \epsilon_1) n_1}{\epsilon_1 m_1 + m_2 + m_{PL}} - \frac{\epsilon_1 (m_1 + m_2 + m_{PL})}{\epsilon_1 m_1 + m_2 + m_{PL}} = \frac{(1 - \epsilon_1) n_1}{1 - \epsilon_2 n_1}$$

⇒

$$\frac{1}{\bar{n}_{PL}} = \frac{m_0}{m_{PL}} = \frac{(1 - \epsilon_1) n_1}{1 - \epsilon_1 n_1} \cdot \frac{(1 - \epsilon_2) n_2}{1 - \epsilon_2 n_2}$$

$f(n_1, n_2)$

$$\frac{m_2 + m_{PL}}{m_{PL}} = \left(\frac{n_2 - 1}{1 - n_2 \epsilon_2} + 1 \right) m_{PL}$$

$$= \frac{n_2 - 1 + 1 - n_2 \epsilon_2}{1 - n_2 \epsilon_2}$$

$$= \frac{(1 - \epsilon_2) n_2}{1 - \epsilon_2 n_2}$$

$$\Rightarrow \therefore \ln \left(\frac{m_0}{m_{PL}} \right) = \ln \left(\frac{(1 - \epsilon_1) n_1}{1 - \epsilon_1 n_1} \right) + \ln \left(\frac{(1 - \epsilon_2) n_2}{1 - \epsilon_2 n_2} \right)$$

$$\Rightarrow \therefore \ln\left(\frac{m_o}{m_{pl}}\right) = \left(\ln(1-\varepsilon_1) + \ln(n_1) - \ln(1-\varepsilon_1 n_1)\right) + \left(\ln(1-\varepsilon_2) + \ln(n_2) - \ln(1-\varepsilon_2 n_2)\right) \Rightarrow f(n_1, n_2)$$

$$\Rightarrow \Delta V_{\text{mission}} = V_{b_o} = \Delta V_1 + \Delta V_2$$

$$\Rightarrow = V_{e1} \ln(n_1) + V_{e2} \ln(n_2) \Rightarrow g(n_1, n_2) = 0$$

11 km/s
for 500 km
orbit

$$\Rightarrow \therefore g(n_1, n_2) = V_{b_o} - V_{e1} \ln(n_1) - V_{e2} \ln(n_2) = 0$$

So, that

$$\Rightarrow h = (n_1, n_2, \eta) = f(n_1, n_2) + \eta g(n_1, n_2)$$

$$\Rightarrow \therefore dh = 0 \Rightarrow \frac{\partial h}{\partial n_1} dn_1 + \frac{\partial h}{\partial n_2} dn_2 + \frac{\partial h}{\partial \eta} d\eta = 0 \quad \star$$

$$\Rightarrow \therefore \left(\text{here } h = [\ln(1-\varepsilon_1) + \ln(n_1) - \ln(1-\varepsilon_1 n_1)] + [\ln(1-\varepsilon_2) + \ln(n_2) - \ln(1-\varepsilon_2 n_2)] + \eta [V_{b_o} - V_{e1} \ln(n_1) - V_{e2} \ln(n_2)] = 0 \right)$$

$$\Rightarrow \therefore \frac{1}{n_1} + \frac{\varepsilon_1}{1-\varepsilon_1 n_1} - \eta \frac{V_{e1}}{n_1} = 0$$

$$\Rightarrow \therefore \frac{1}{n_2} + \frac{\varepsilon_2}{1-\varepsilon_2 n_2} - \eta \frac{V_{e2}}{n_2} = 0$$

$$\Rightarrow \therefore V_{b_o} - V_{e1} \ln(n_1) - V_{e2} \ln(n_2) = 0$$

$$\text{--- } ① \Rightarrow n_1 = \text{--- } ①, ②, ③$$

RHS is zero
because of $\star$ equation

$$\text{--- } ② \Rightarrow n_2 = \text{---}$$

RHS is zero
each term has to be 0

$$\text{--- } ③$$

$$\therefore \text{for } n_1 \Rightarrow \frac{1}{n_1} (1 - n V_{e1}) = \frac{-\epsilon_1}{1 - \epsilon_1 n_1}$$

$$\Rightarrow \frac{1 - \epsilon_1 n_1}{n_1} = \frac{-\epsilon_1}{1 - n V_{e1}}$$

$$\Rightarrow \frac{1}{n_1} - \epsilon_1 = \frac{-\epsilon_1}{1 - n V_{e1}}$$

$$\Rightarrow \frac{1}{n_1} = \frac{-\epsilon_1}{1 - n V_{e1}} + \epsilon_1 = \frac{-\epsilon_1 + \epsilon_1 - \epsilon_1 n V_{e1}}{1 - n V_{e1}}$$

$$\Rightarrow \boxed{n_1 = \frac{n V_{e1} - 1}{\epsilon_1 n V_{e1}}}$$

Similarly,

$$\Rightarrow n_2 = \frac{V_{e2} n - 1}{V_{e2} \epsilon_2 n}$$

$$\Rightarrow V_{bo} = V_{e1} \ln n_1 + V_{e2} \ln n_2$$

$$\Rightarrow \therefore \boxed{V_{bo} = V_{e1} \ln \left(\frac{V_{e1} n - 1}{V_{e1} \epsilon_1 n} \right) + V_{e2} \ln \left(\frac{V_{e2} n - 1}{V_{e2} \epsilon_2 n} \right) + \underbrace{\frac{-V_{e3} \ln (V_{e3} n - 1)}{V_{e3} \epsilon_3 n}}_{\text{for more stages}} + V_{e0}}$$

from this equation we need to solve for n numerically once we get n , we will be able find n_1, n_2 .

⇒ Let, we want to minimize the function h , then

⇒ $d^2 h = +Ve$

⇒ $n V_{ei} (\epsilon_i n_i - 1)^2 + 2 \epsilon_i n_i - 1 > 0$

for each Stage ensuring it as local minimum

N-Stage Vehicle

⇒
$$\sum_{i=1}^N V_{ei} \ln(V_{ei} n_i - 1) - \ln\left(n_i \sum_{i=1}^N V_{ei}\right) - \sum_{i=1}^N V_{ei} \ln(V_{ei} \epsilon_i) = V_{b0}$$

⇒
$$n_i = \frac{V_{ei} n_i - 1}{V_{ei} \epsilon_i \cdot n_i}$$

⇒
$$\sum_{i=1}^N V_{ei} \ln \frac{V_{ei} n_i - 1}{V_{ei} \epsilon_i \cdot n_i} = V_{b0}$$

⇒
$$m_{N-1} \dots m_1 = \left(\frac{(n-1)}{(1 - n_i \epsilon_i)} \right) (m_2 + m_3 + \dots m_{pL})$$

⇒
$$m_N = \frac{n_N - 1}{1 - m_N \epsilon_N} \cdot m_{pL}$$

N-Stage Vehicle : Continued

Date : 22/01/2026

Numerical example :- 3-Stage launch Vehicle :

$$\Rightarrow m_{PL} = 500 \text{ kg}, \Delta V_{\text{mission}} = 10 \text{ km/s}$$

$$\Rightarrow \text{Stage 1} \rightarrow I_{sp} = 400 \text{ s} \rightarrow 3.924 \text{ km/s} (\epsilon_1 = 0.10)$$

$$\Rightarrow \text{Stage 2} \rightarrow I_{sp} = 350 \text{ s} \rightarrow 3.434 \text{ km/s} (\epsilon_2 = 0.15)$$

$$\Rightarrow \text{Stage 3} \rightarrow I_{sp} = 300 \text{ s} \rightarrow 2.943 \text{ km/s} (\epsilon_3 = 0.20)$$

find $m_0, m_{p1}, m_{p2}, m_{p3}, m_{E1}, m_{E2}, m_{E3}$

$$\Rightarrow V_{e1} \ln \left(\frac{V_{e1} n - 1}{V_{e1} \epsilon_1 n} \right) + V_{e2} \ln \left(\frac{V_{e2} n - 1}{V_{e2} \epsilon_2 n} \right) + V_{e3} \ln \left(\frac{V_{e3} n - 1}{V_{e3} \epsilon_3 n} \right) = V_{b0} \Delta V_{\text{mission}}$$

Numerically Solving for n

$$\Rightarrow 3.924 \ln \left(\frac{3.924 n - 1}{3.924 \times 0.1 n} \right) + 3.434 \ln \left(\frac{3.434 n - 1}{3.434 \times 0.15 n} \right) + 2.943 \ln \left(\frac{2.943 n - 1}{2.943 \times 0.2 n} \right)$$

$$= 10 \text{ km/s}$$

$\therefore$ Value of n is coming out to be $n = 0.4668$

$$\Rightarrow n_1 = \frac{V_{e1} n - 1}{V_{e1} \epsilon_1 n}$$

$$\Rightarrow n_1 = \frac{3.924 \times 0.4668 - 1}{3.924 \times 0.1 \times 0.4668}$$

$$\Rightarrow n_1 = 4.54$$

$$m_3 = \frac{n_N - 1}{1 - n_N \epsilon_N} \cdot m_{PL}$$

$$n_3 = \frac{V_{e3} n - 1}{V_{e3} \epsilon_3 n} = \frac{2.943 \times 0.4668 - 1}{2.943 \times 0.4668 \times 0.2} = 1.3604$$

$$n_2 = 2.5077$$

$$m_3 = \frac{n_3 - 1}{1 - n_3 \epsilon_3} m_{PL} = 2475.5 \text{ kg}$$

$$m_2 = \frac{n_2 - 1}{1 - n_2 \epsilon_2} \cdot (m_3 + m_{PL}) = 18066.6 \text{ kg}$$

$$m_1 = \frac{n_1 - 1}{1 - n_1 \epsilon_1} (m_2 + m_3 + m_{PL}) = 165602.6 \text{ kg}$$

$$m_0 = m_1 + m_2 + m_3 + m_{PL} = 191144.7 \text{ kg}$$

$$\pi_{PL} = \frac{m_{PL}}{m_0} = 0.02615 \approx 2\%$$

$$m_{01} = m_1 + m_2$$

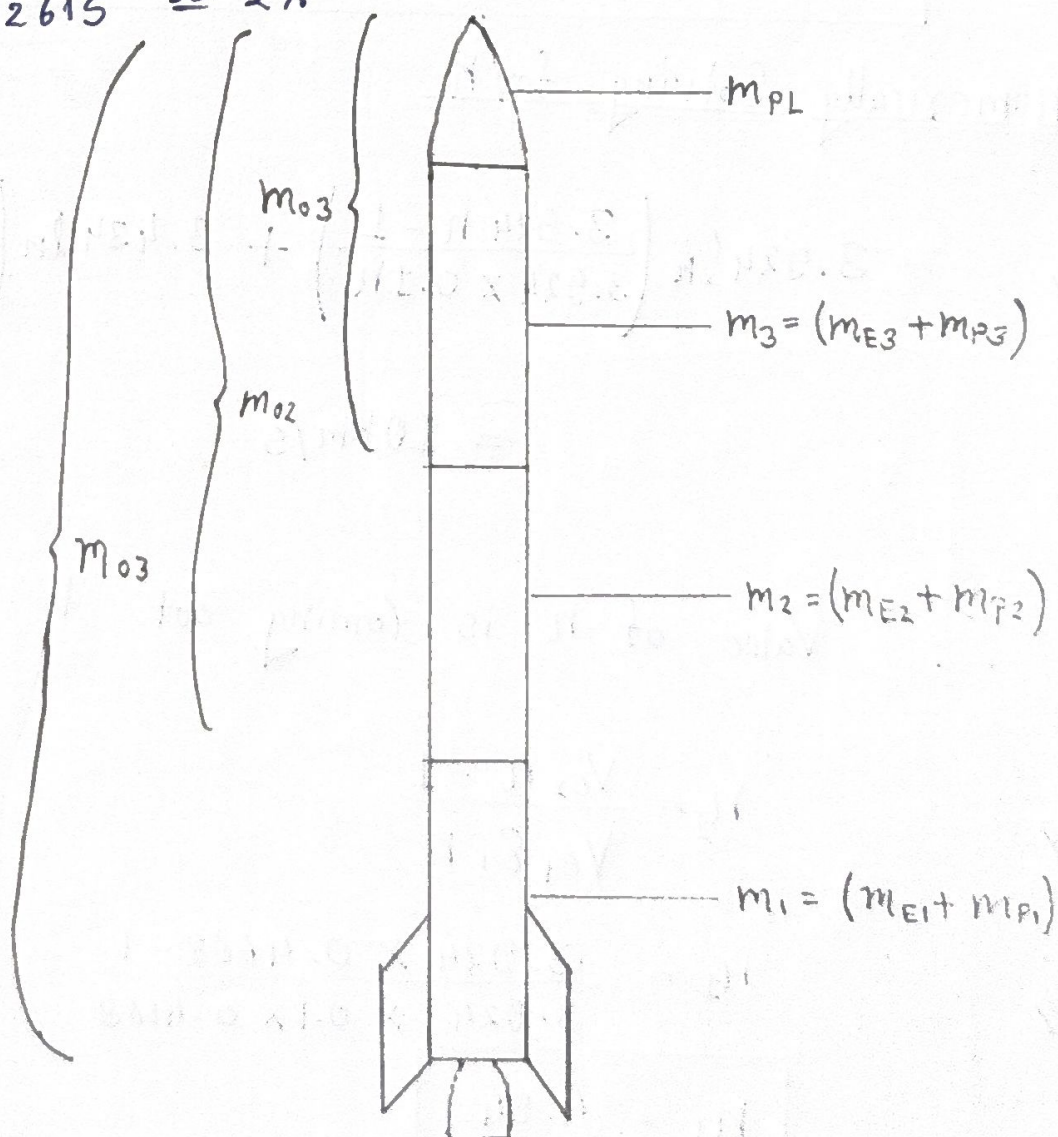
$$m_{02} = m_2 + m_3$$

$$m_{03} = m_3 + m_{PL}$$

$$\pi_{PL1} = \frac{m_{02}}{m_{01}}$$

$$\pi_{PL2} = \frac{m_{03}}{m_{02}}$$

$$\pi_{PL3} = \frac{m_{PL}}{m_{03}}$$



$$\Delta V = V_e \ln(n)$$

$$\Rightarrow \eta = \frac{m_p + m_E + m_{PL}}{m_E + m_{PL}} \rightarrow \eta = \frac{1 + \lambda}{\epsilon + \lambda}$$

$$\Rightarrow \lambda = \frac{m_{PL}}{m_0 - m_{PL}} \Rightarrow \frac{1}{\frac{m_0}{m_{PL}} - 1} \Rightarrow \frac{\pi_{PL}}{1 - \pi_{PL}}$$

$$\Rightarrow \eta_1 = \frac{1}{\pi_{PL}^{1/2} (1 - \epsilon) + \epsilon}$$

$$\Rightarrow \Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3 = V_{e1} \ln(n_1) + V_{e2} \ln(n_2) + V_{e3} \ln(n_3)$$

$$\Rightarrow \Delta V = V_{e1} \ln\left(\frac{1}{\epsilon_1 + \pi_{PL1}(1 - \epsilon_1)}\right) + V_{e2} \ln\left(\frac{1}{\epsilon_2 + \pi_{PL2}(1 - \epsilon_2)}\right) + V_{e3} \ln\left(\frac{1}{\epsilon_3 + \pi_{PL3}(1 - \epsilon_3)}\right)$$

$\Rightarrow$ π_{PL} is different than π_{PL1}

$$\Rightarrow \text{as } \pi_{PL} = \frac{m_{PL}}{m_0} \text{ \& } \pi_{PL1} = \frac{m_{02}}{m_{01}}$$

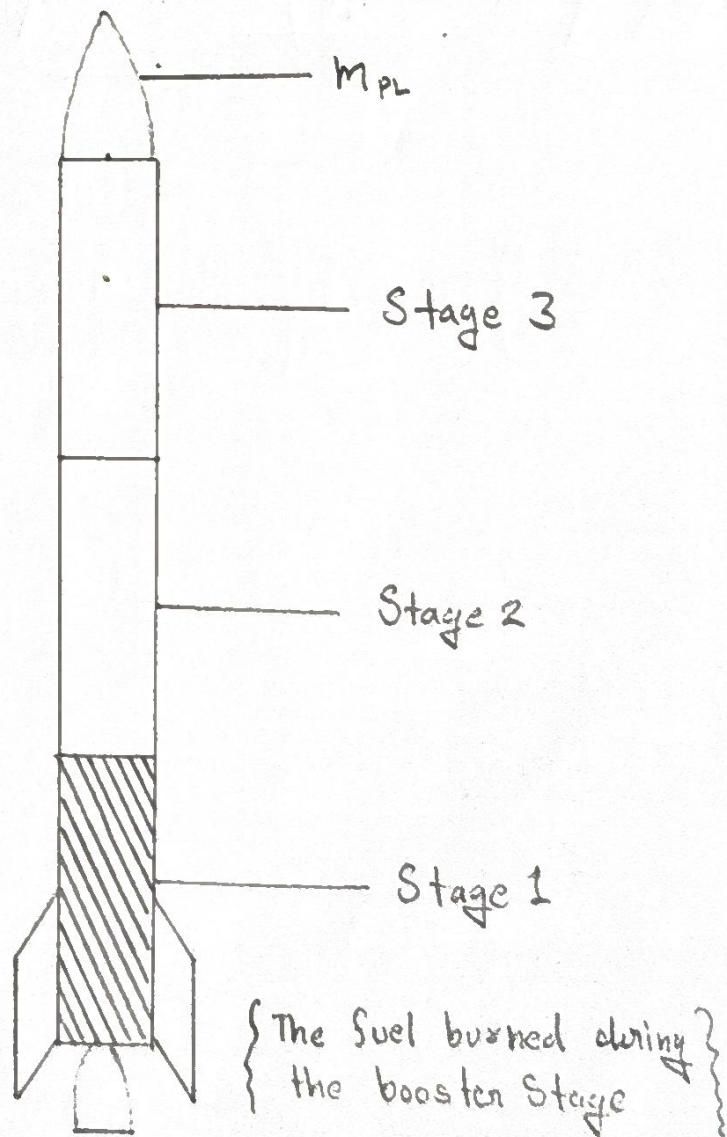
$$\Rightarrow \Delta V_{\text{total - no booster}} = \Delta V_1 + \Delta V_2 + \Delta V_3 = V_{e1} \ln(n_1) + V_{e2} \ln(n_2) +$$

$$\Rightarrow m_{03} = \frac{m_{PL}}{\pi_{PL3}}$$

$$\Rightarrow m_{E3} = \epsilon_3 (m_{03} - m_{PL})$$

$$\Rightarrow m_{p3} = m_{03} - (m_{E3} + m_{PL})$$

$$\Rightarrow m_{02} = \frac{m_{03}}{\pi_{PL2}}$$



$$\Rightarrow m_{E2} = \varepsilon_2 (m_{o2} - m_{o3})$$

$$\Rightarrow m_{p2} = m_{o2} - (m_{E2} + m_{o3})$$

$$\Rightarrow m_{o1} = \frac{m_{o2}}{\kappa_{PL1}}$$

$$\Rightarrow m_{E1} = \varepsilon_1 (m_{o1} - m_{o2})$$

$$\Rightarrow m_{p1} = m_{o1} (m_{E1} + m_{o2})$$

$$\Rightarrow m_{pb} = \text{booster propellant mass}$$

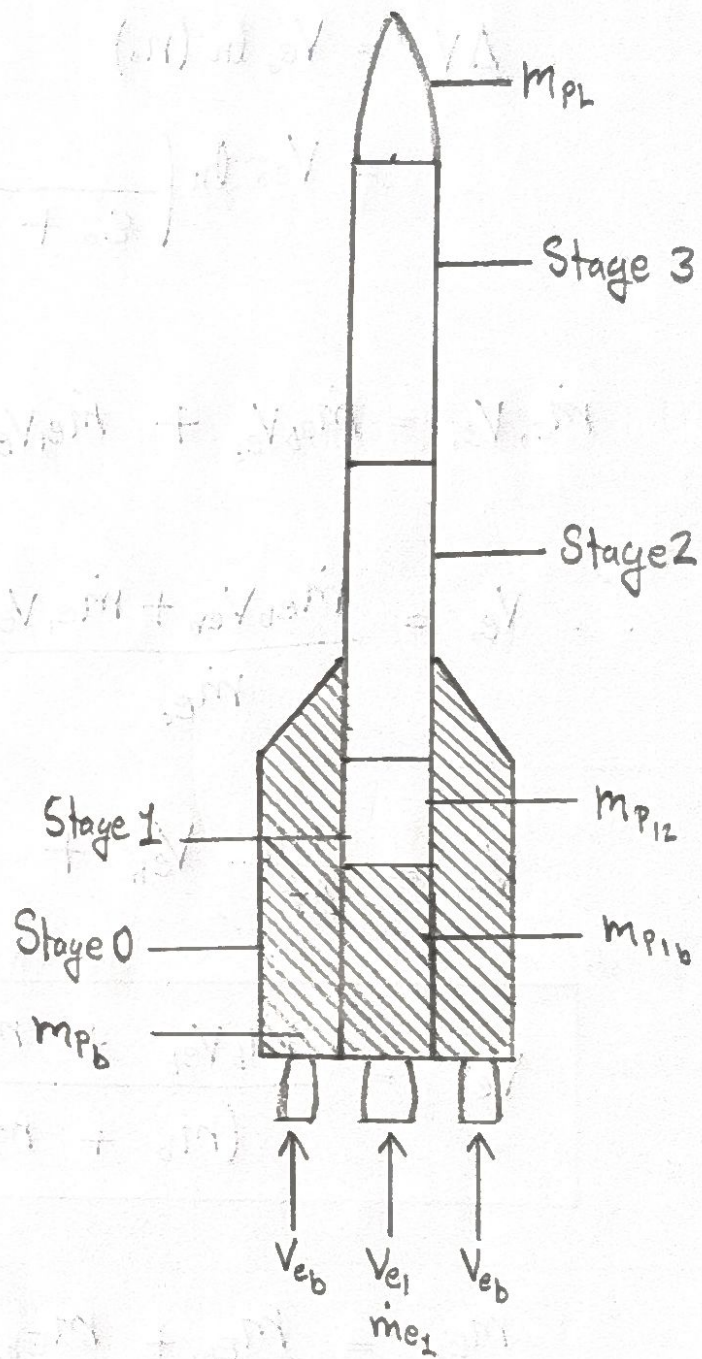
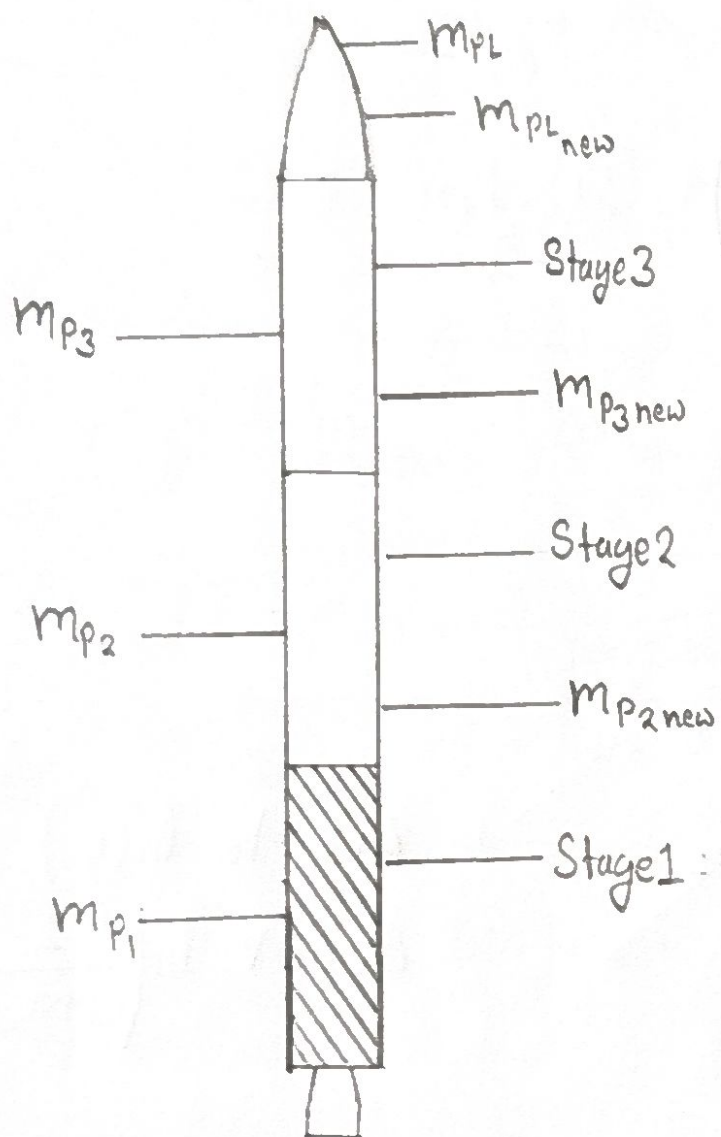
$$\Rightarrow m_{p1} \rightarrow \text{Stage 1 (Solid/Liquid)}$$

$\left. \begin{array}{l} \text{Propellant mass consumed along with booster strap mass} \end{array} \right\}$

$$\Rightarrow I_{sp_b} \text{ — Typically } \sim 200s \text{ to } \sim 300s$$

N-Stage Vehicle : Continued

Date : 23/01/2026



Zeroth Stage :-

π_{PL0}

$$\Delta V_{\text{total boosters}} = \Delta V_0 + \Delta V_1 + \Delta V_2 + \Delta V_3$$

$\Rightarrow$

$$\Delta V_{\text{total no-boosters}} = \Delta V_1 + \Delta V_2 + \Delta V_3$$

$\Rightarrow$

$$\pi_{PL0} = \frac{m_{PL0}}{m_{00}} \quad (m_{00} = m_{01} + m_{P6} + m_{E6})$$

$\Rightarrow$

$$\Rightarrow \epsilon_o = \frac{m_{Eb} + m_{E1}}{m_{Eb} + m_{E1} + m_{Pb} + m_{P10}}$$

$$\Rightarrow \Delta V_o = V_o \ln(n_o)$$

$$\Rightarrow = V_o \ln\left(\frac{1}{\epsilon_o + \pi_{Pb}(1-\epsilon_o)}\right)$$

$$\Rightarrow \dot{m}_o V_o = \dot{m}_{Eb} V_{Eb} + \dot{m}_{E1} V_{E1}$$

$$\Rightarrow \therefore V_o = \frac{\dot{m}_{Eb} V_{Eb} + \dot{m}_{E1} V_{E1}}{\dot{m}_o}$$

$$\Rightarrow V_o = \frac{m_b}{\Delta t} V_{Eb} + \frac{m_{P10}}{\Delta t} V_{E1}$$

$$\Rightarrow V_o = \frac{m_b V_{Eb} + m_{P1} V_{E1}}{(m_b + m_{P10})}$$

$$\Rightarrow m_{Eb} = m_{Eb1} + m_{Eb2}$$

$$\Rightarrow m_{Eb1} = m_{Eb2}$$

$$\Rightarrow m_{Eb} = 2 \times m_{Eb1}$$

1st Stage

$$\Rightarrow \epsilon'_1 = \frac{m_{E1}}{m_{E1} + m_{b1}}$$

$$\Delta V_o = V_o \ln(n_o)$$

$$\Delta V_o = V_o \ln\left(\frac{1}{\epsilon_o + \pi_{Pb}(1-\epsilon_o)}\right)$$

$$\Rightarrow m_{p_{11}} = m_{p_1} - m_{p_{10}}$$

$$\Rightarrow \pi'_{p_{L_1}} = \frac{m_{02}}{m_{01} + m_{p_{10}}}$$

$$\Rightarrow \Delta V_1' = V_{e1} \ln(n_1') = V_{e1} \ln\left(\frac{1}{\epsilon_1' + \pi_1'(1-\epsilon_1')}\right)$$

$$\Rightarrow \boxed{\Delta V_{\text{total no-booster}} \leq \Delta V_{\text{total booster}}}$$

let's assume 12 km/s

Booster

$$\Rightarrow \Delta_{\text{total booster}} = \Delta V_0 + \Delta V_1 + \Delta V_2 + \Delta V_{3\text{new}}$$

{ $\Delta V_{3\text{new}}$ has to be found out }

$$\Rightarrow \underbrace{\Delta V_{3\text{new}}}_{\text{less than } \Delta V_3} = \Delta V_{\text{total booster}} - \Delta V_0 - \Delta V_1 - \Delta V_2$$

$$\Rightarrow \Delta V_{3\text{new}} = V_{e3} \ln n_{3\text{new}}$$

$$\Rightarrow m_{03} = m_{E3} + m_{p_{3\text{new}}} + m_{p_{L\text{new}}}$$

$$\Rightarrow \boxed{\Delta V_{3\text{new}} = V_{e3} \ln\left(\frac{m_{03}}{m_{03} - m_{p_{3\text{new}}}}\right)}$$

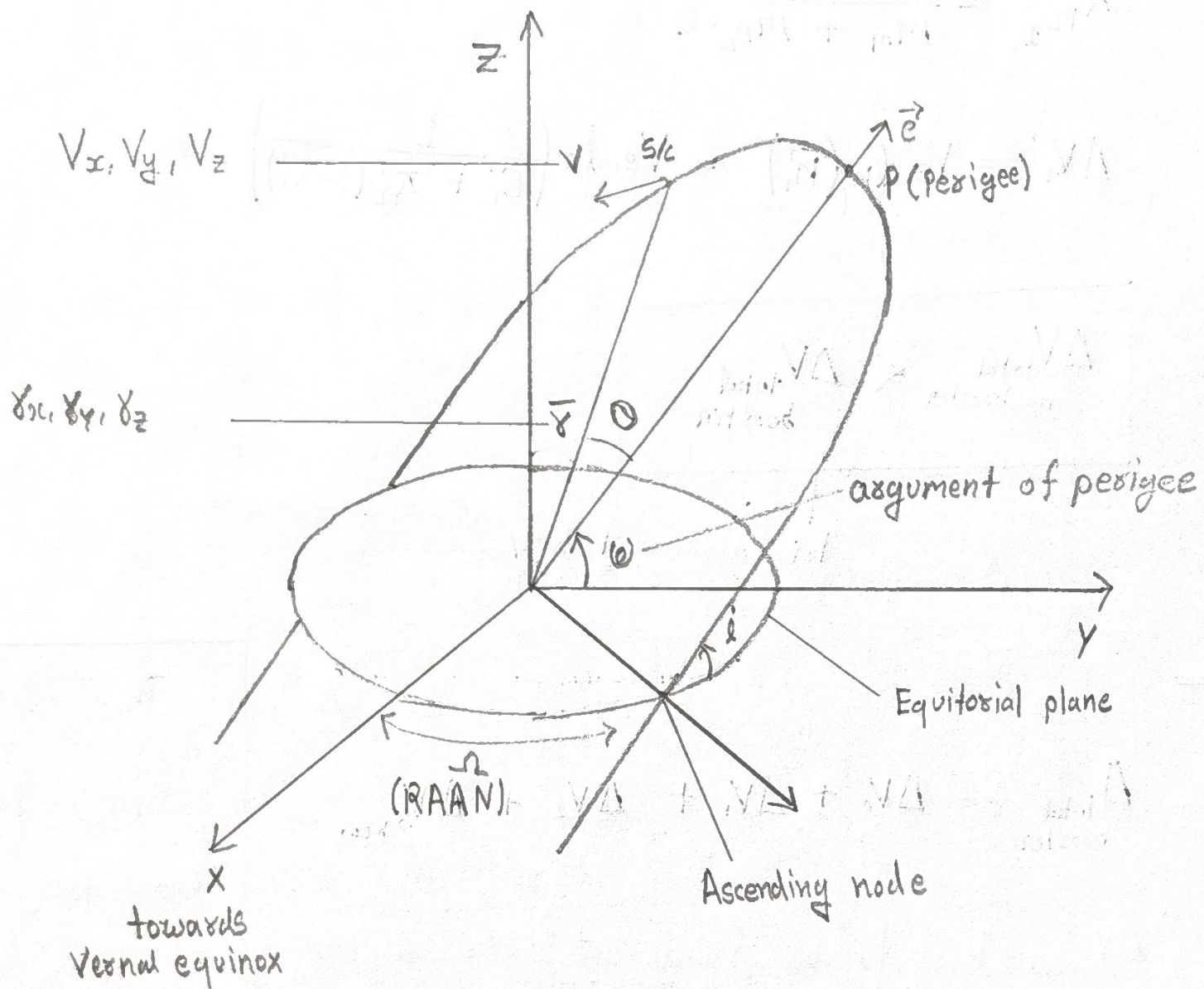
{ find m_{p_3} new then find $m_{p_{L\text{new}}}$ }

$$m_{03} = \text{const.}$$

So, $m_{03} = m_{03\text{new}}$

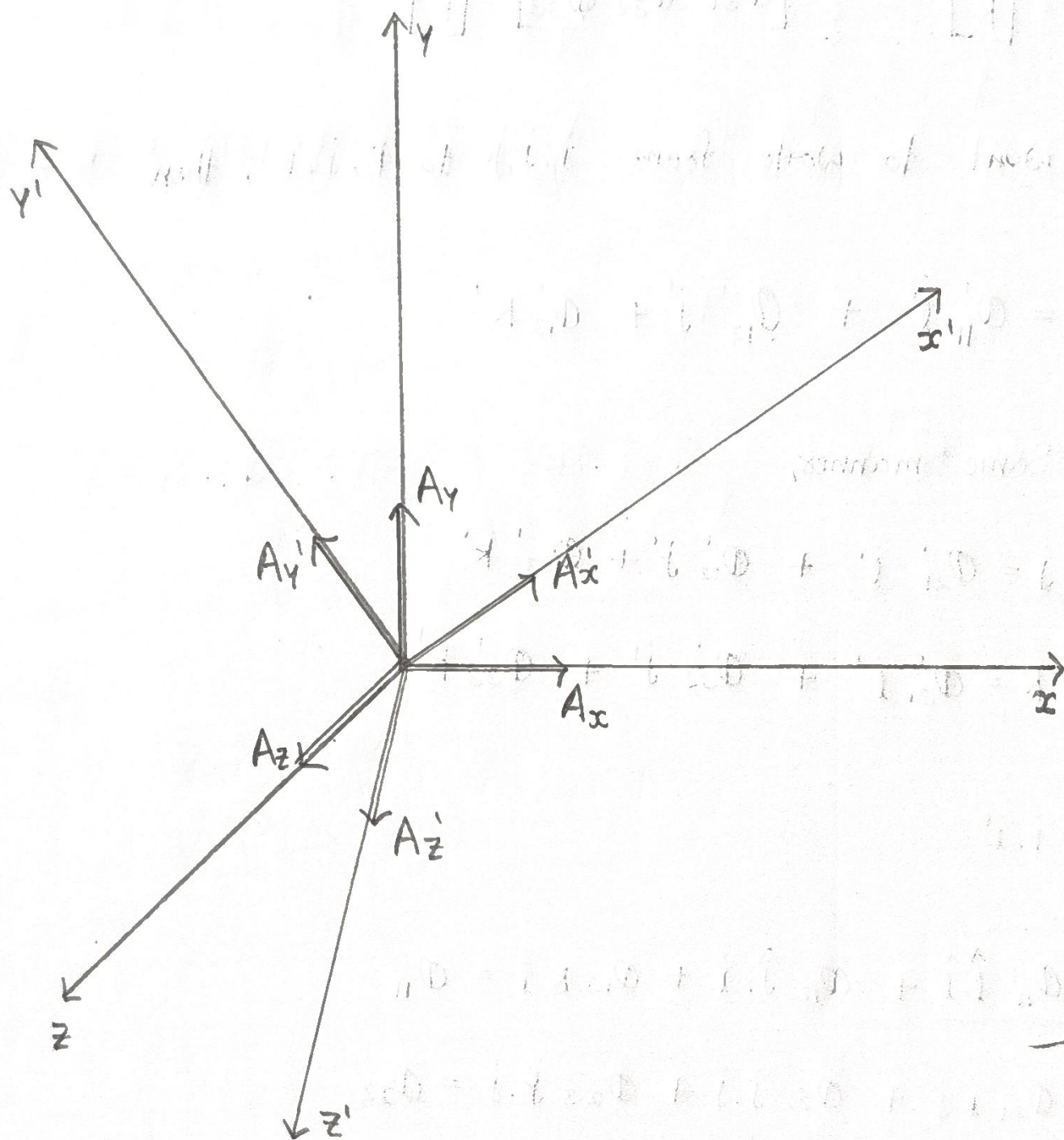
Our Strong assumption

$$m_{E3} = m_{E3\text{new}}$$



Co-ordinate Transformation (Ref. Curtis)

Date: 29/01/2026



$$\Rightarrow \hat{i}' = \mathcal{O}_{11} \hat{i} + \mathcal{O}_{21} \hat{j} + \mathcal{O}_{31} \hat{k}$$

$\left\{ \begin{array}{l} \text{Projection of} \\ \hat{i}' \text{ on } \hat{i} \end{array} \right\}$
 $\left\{ \begin{array}{l} \text{Projection} \\ \text{of } \hat{i}' \text{ on } \hat{j} \end{array} \right\}$
 $\left\{ \begin{array}{l} \text{Projection} \\ \text{of } \hat{i}' \text{ on } \hat{k} \end{array} \right\}$

Similarly,

$$\Rightarrow \hat{j}' = \mathcal{O}_{21} \hat{i} + \mathcal{O}_{22} \hat{j} + \mathcal{O}_{23} \hat{k}$$

$$\Rightarrow \hat{k}' = \mathcal{O}_{31} \hat{i} + \mathcal{O}_{32} \hat{j} + \mathcal{O}_{33} \hat{k}$$

$$\Rightarrow \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

if we want to write from i, j, k to i', j', k' , then

$$\Rightarrow i = Q'_{11} i' + Q'_{12} j' + Q'_{13} k'$$

In the same manner,

$$\Rightarrow j = Q'_{21} i' + Q'_{22} j' + Q'_{23} k'$$

$$\Rightarrow k = Q'_{31} i' + Q'_{32} j' + Q'_{33} k'$$

as $i' \cdot i = i \cdot i'$

$$\Rightarrow i' \cdot i = \underline{Q'_{11} \hat{i} \cdot \hat{i}} + Q'_{12} \hat{j} \cdot \hat{i} + Q'_{13} \hat{k} \cdot \hat{i} = Q'_{11}$$

$$\Rightarrow j' \cdot j = Q'_{21} \hat{i} \cdot \hat{j} + \underline{Q'_{22} \hat{j} \cdot \hat{j}} + Q'_{23} \hat{k} \cdot \hat{j} = Q'_{22}$$

$$\Rightarrow k' \cdot k = Q'_{31} \hat{i} \cdot \hat{k} + Q'_{32} \hat{j} \cdot \hat{k} + \underline{Q'_{33} \hat{k} \cdot \hat{k}} = Q'_{33}$$

Similarly,

$$\Rightarrow i \cdot i' = Q'_{11} i' \cdot i' + Q'_{12} j' \cdot i' + Q'_{13} k' \cdot i'$$

We can write the same for $i', j, j \cdot i'$ etc.

$$\Rightarrow Q' = Q^T = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{13} \\ Q'_{12} & Q'_{22} & Q'_{23} \\ Q'_{13} & Q'_{23} & Q'_{33} \end{bmatrix}$$

diagonal elements will be same as Q

Rest of the elements will be 0 at the cross entries.

$$\Rightarrow \left. \begin{aligned} Q^T \cdot Q &= I \\ Q \cdot Q^T &= I \end{aligned} \right\} \text{Orthogonality of } Q$$

for first Coordinate System

$$\Rightarrow V = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$$

for Second

$$\Rightarrow V = V_{x'} \hat{i}' + V_{y'} \hat{j}' + V_{z'} \hat{k}'$$

if we want find V along $\hat{i}'$, the very basic step will be fundamentally

$$\Rightarrow V \cdot \hat{i} \rightarrow V_{x'}$$

$$\Rightarrow V \cdot \hat{j} \rightarrow V_{y'}$$

$$\Rightarrow V \cdot \hat{k} \rightarrow V_{z'}$$

$$\Rightarrow V_{x'} \hat{i}' + V_{y'} \hat{j}' + V_{z'} \hat{k}' = V_x \hat{i} + V_y \hat{j} + V_z \hat{k} \quad \left. \begin{aligned} &\text{\{not changed\}} \\ &\therefore V = V \end{aligned} \right\}$$

$$\Rightarrow \hat{i} = Q_{11} \hat{i}' + Q_{12} \hat{j}' + Q_{13} \hat{k}'$$

$$\Rightarrow \hat{j} = Q_{21} \hat{i}' + Q_{22} \hat{j}' + Q_{23} \hat{k}'$$

$$\Rightarrow \hat{k} = Q_{31} \hat{i}' + Q_{32} \hat{j}' + Q_{33} \hat{k}'$$

$$\Rightarrow V_{x'} = V_x Q_{11} + V_y Q_{12} + V_z Q_{13}$$

$$\Rightarrow V_{y'} = V_x Q_{21} + V_y Q_{22} + V_z Q_{23}$$

$$\Rightarrow V_{z'} = V_x Q_{31} + V_y Q_{32} + V_z Q_{33}$$

$$\Rightarrow \{v'\} = [Q] \{v\}$$

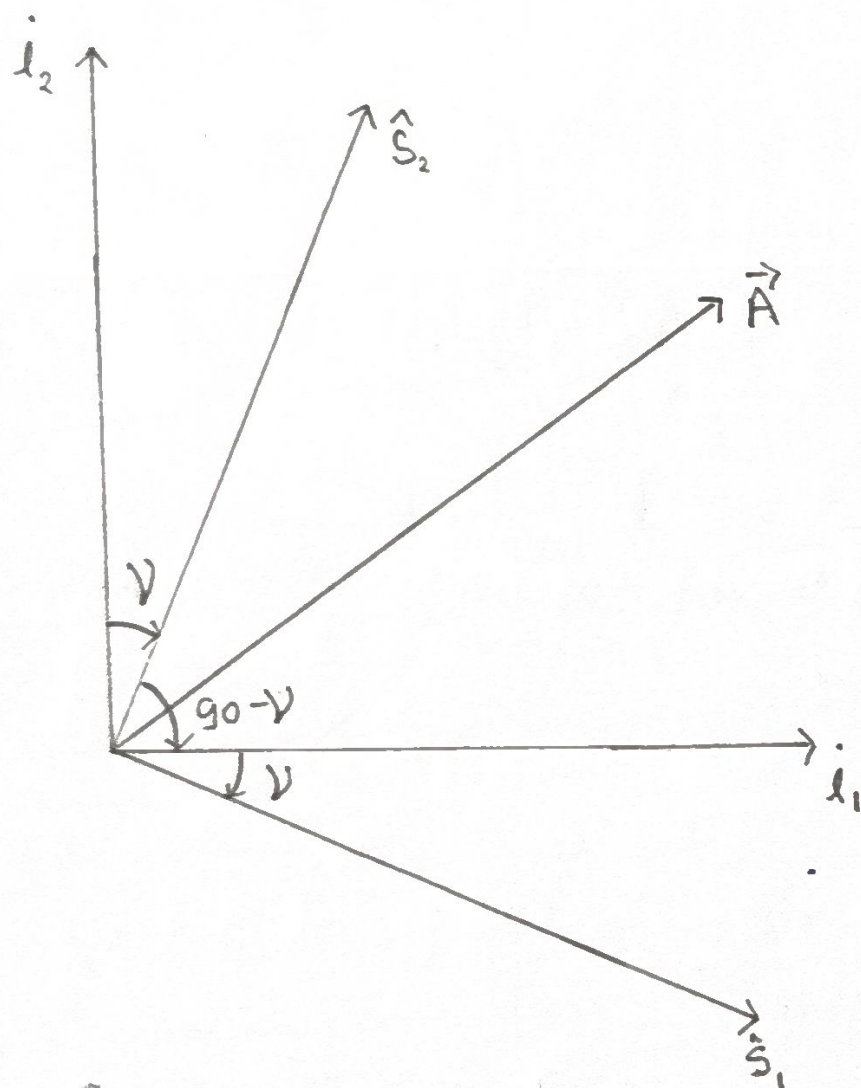
$$\Rightarrow Q^T \{v'\} = \underbrace{Q^T Q}_I \{v\} \quad \{v\} = Q \{v'\} \quad \{v'\} = Q^T \{v\}$$

Date :- 03/02/2026 : Coordinate Transformation Involving Rotations

- Rotation about x -axis $\Rightarrow R_1 (\quad)$
- Rotation about y -axis $\Rightarrow R_2 (\quad)$
- Rotation about z -axis $\Rightarrow R_3 (\quad)$

R_1, R_2, R_3

Rotation Seq not
not necessarily in the
order R_1, R_2, R_3



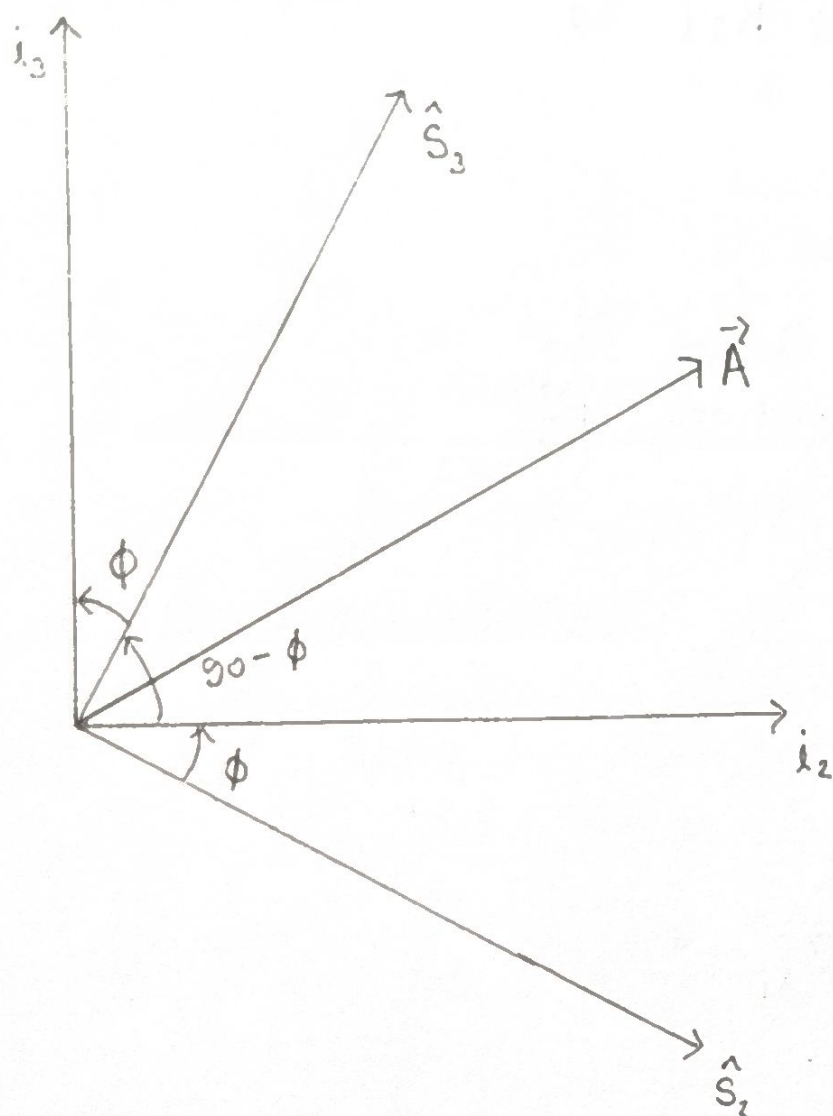
Curtis
(α, β, γ)

$$\Rightarrow \vec{A} = A_{S1} \hat{S}_1 + A_{S2} \hat{S}_2 + A_{S3} \hat{S}_3$$

Rotation with respect. to $\hat{i}_3, \hat{S}_3 \rightarrow z$

$$\begin{aligned} \Rightarrow A_{i1} &= \vec{A} \cdot \hat{i}_1 \\ \Rightarrow A_{i2} &= \vec{A} \cdot \hat{i}_2 \\ \Rightarrow A_{i3} &= \vec{A} \cdot \hat{i}_3 \end{aligned} \quad \begin{bmatrix} i_{S1} & i_{S2} & i_{S3} \\ i_{2S1} & i_{2S2} & i_{2S3} \\ i_{3S1} & i_{3S2} & i_{3S3} \end{bmatrix} \quad \begin{bmatrix} A_{S1} \\ A_{S2} \\ A_{S3} \end{bmatrix}$$

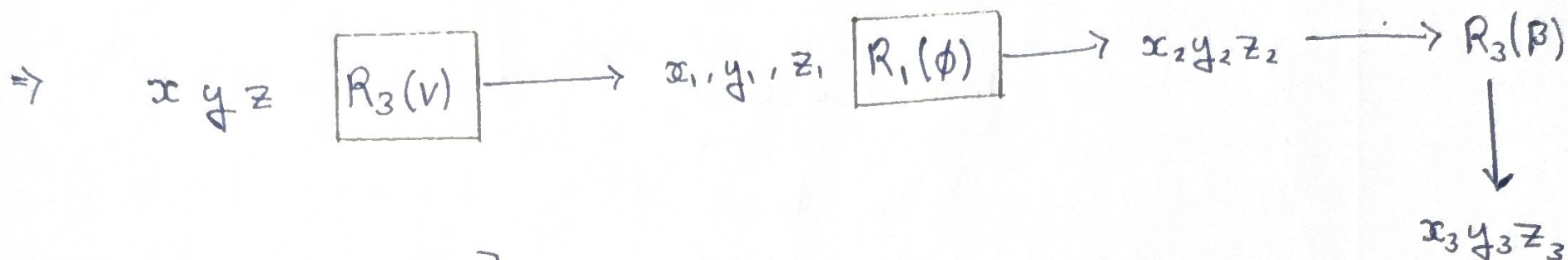
$$\Rightarrow \begin{bmatrix} A_{i_1} \\ A_{i_2} \\ A_{i_3} \end{bmatrix} = \begin{bmatrix} \cos \nu & \cos(90-\nu) & \cos 90^\circ \\ \cos(90+\nu) & \cos \nu & \cos 90^\circ \\ \cos 90^\circ & \cos 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} \cos \nu & \sin \nu & 0 \\ -\sin \nu & \cos \nu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\Rightarrow \begin{bmatrix} A_{i_1} \\ A_{i_2} \\ A_{i_3} \end{bmatrix} = \begin{bmatrix} \cos 0^\circ & \cos 90^\circ & \cos 90^\circ \\ \cos 90^\circ & \cos \phi & \cos(90-\phi) \\ \cos 90^\circ & \cos(90+\phi) & \cos \phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix}$$

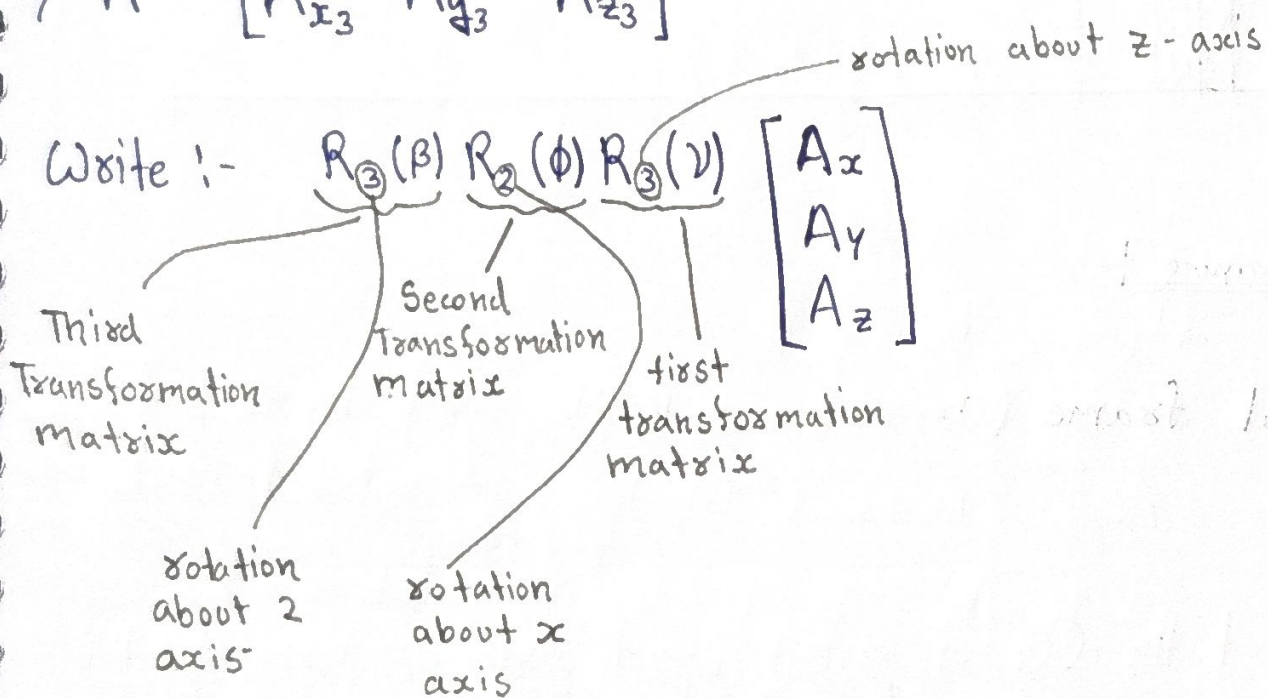
$\Rightarrow$ Similarly for y-axis :-

$$\Rightarrow \begin{bmatrix} A_{i1} \\ A_{i2} \\ A_{i3} \end{bmatrix} = \begin{bmatrix} \cos \psi & \cos 90^\circ & -\sin \psi \\ \cos 90^\circ & \cos 0^\circ & \cos 90^\circ \\ \sin \psi & \cos 90^\circ & \cos \psi \end{bmatrix} = \begin{bmatrix} \cos \psi & 0 & -\sin \psi \\ 0 & 1 & 0 \\ \sin \psi & 0 & \cos \psi \end{bmatrix}$$



$$\Rightarrow \vec{A} = [A_x \ A_y \ A_z]$$

$$\Rightarrow \vec{A} = [A_{x3} \ A_{y3} \ A_{z3}]$$

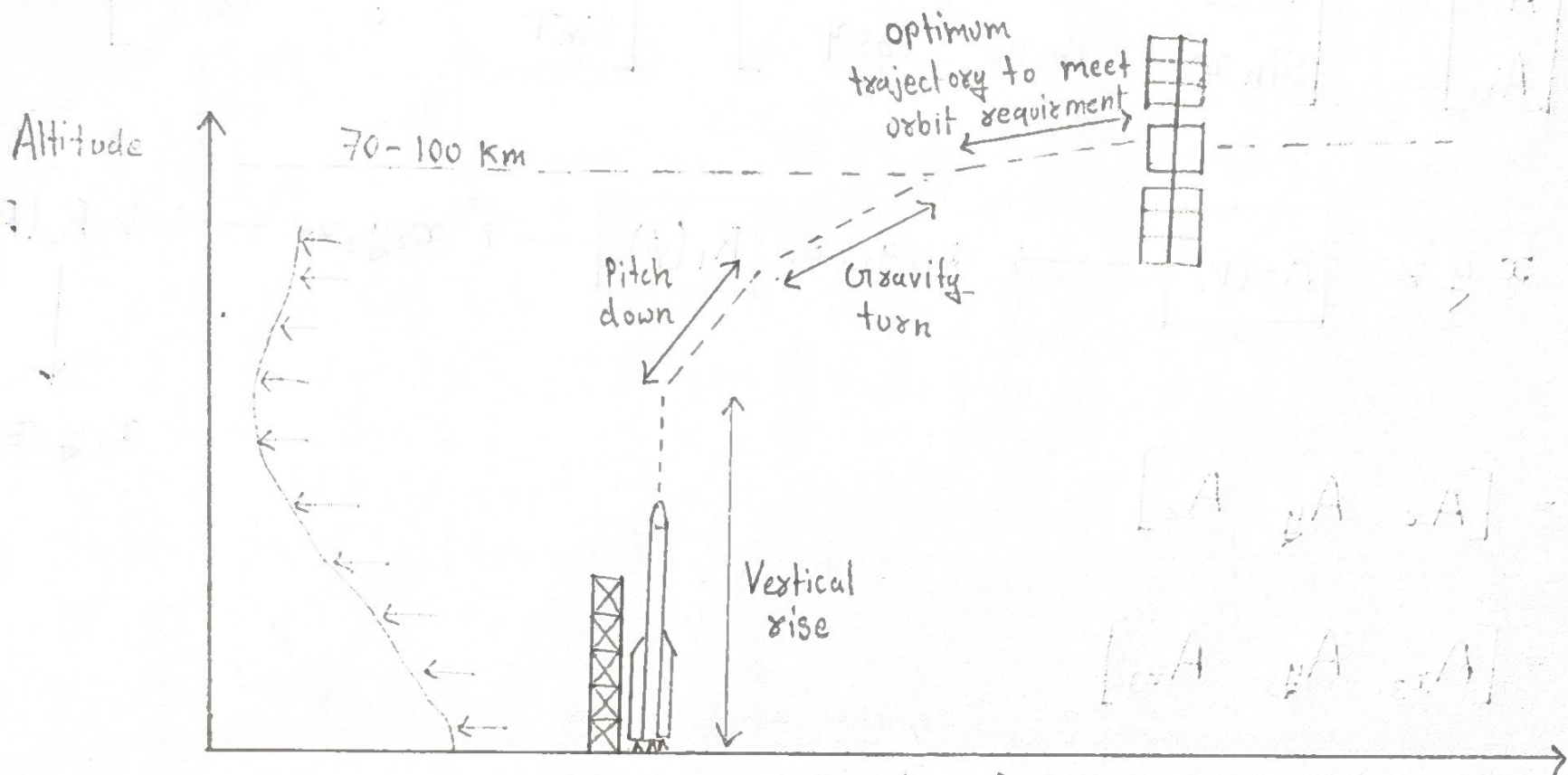


$\Rightarrow$ last operation come first

Assignment :- Satellite to be place at particular, trajectory optimizer (Sizing also)

Launch Trajectory

Date: - 04/02/2026

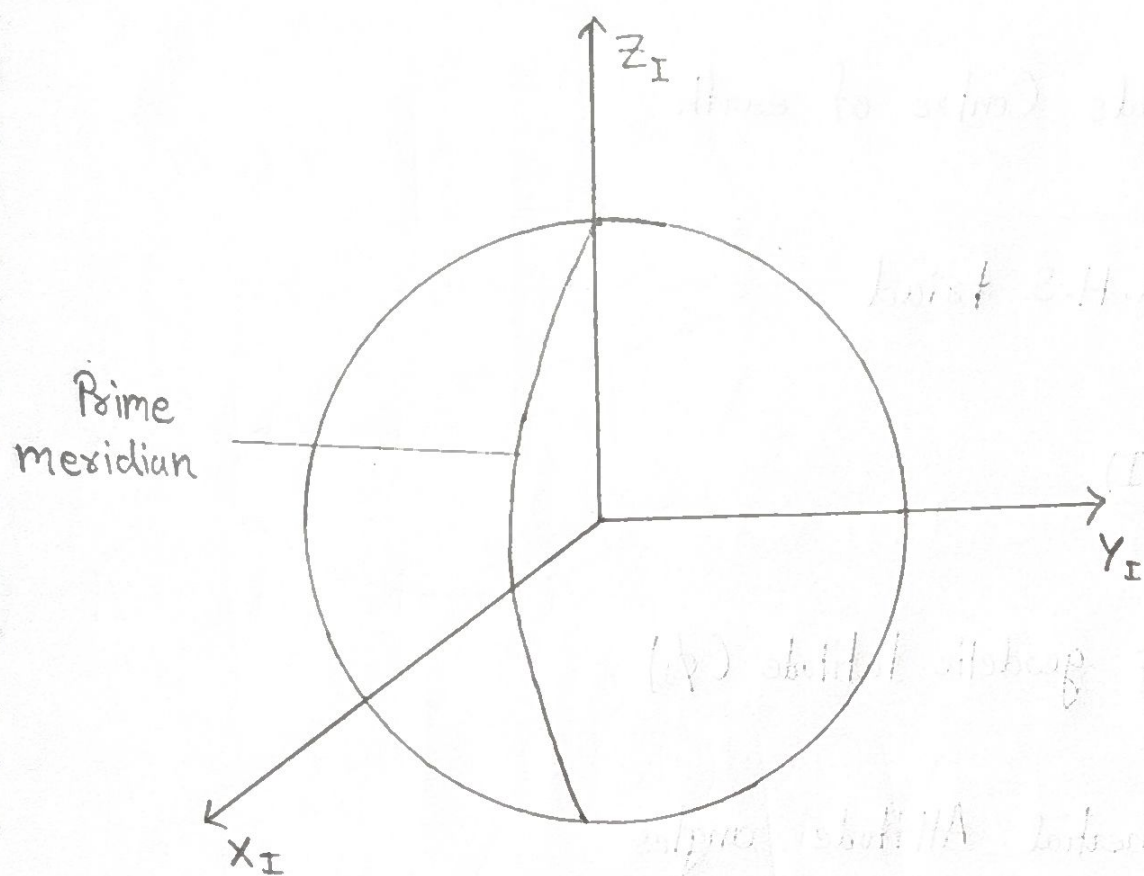


Names of Reference frames :-

- 1 $\Rightarrow$ Launch pad Inertial frame (Orientation of Vehicle w.r.t launch pad Inertial frame)
- 2 $\Rightarrow$ Geograph frame :- Connect the C.G. of rocket w.r.t earth centered inertial frame (from the centre of earth) project that vector on the equatorial plane.
- 3 $\Rightarrow$ Aerodynamic relative Velocity frame
- 4 $\Rightarrow$ Earth Centered Inertial
- 5 $\Rightarrow$ Body frame
- 6 $\Rightarrow$ Earth Centered rotating frame.

Coordinate frames

1 $\Rightarrow$ ECI - Earth Centered Inertial frame of reference.



$\rightarrow X_I$ - along the Greenwich meridian at $t = 0 \rightarrow$ remains fixed
 $\Rightarrow$ will not change (move) along with earth.

$\rightarrow Z_I$ - towards north pole

$\rightarrow Y_I$ - Right hand thumb rule

2 $\Rightarrow$ ECR - Earth Centered rotating frame :-

$\rightarrow X_R$ - will rotate along with earth (earth centered earth fixed)

$\rightarrow Y_R$ - Right hand rule

$\rightarrow Z_R$ - north pole

3 $\Rightarrow$ Geographic frame (LV-LH) $\Rightarrow$ local vertical, local horizontal)

X_{cr} = local north on the horizon

Z_{cr} = point towards Centre of earth

Y_{cr} = Completes R.H.S triad

4 $\Rightarrow$ Inertial Launch frame (LPI)

LPI $\rightarrow$ located at geodetic latitude (ϕ_d)

Inertial References frame $\rightarrow$ Inertial Attitude angles
(θ_{pitch} , ψ_{yaw} , ϕ_{roll})

A_{Z_L} $\rightarrow$ launch azimuth $\rightarrow$ Angles measured Clockwise from local north of launch site to the projected direction of trajectory on the surface of earth.

ϕ_d = geodetic latitude

θ_L = geocentric longitude

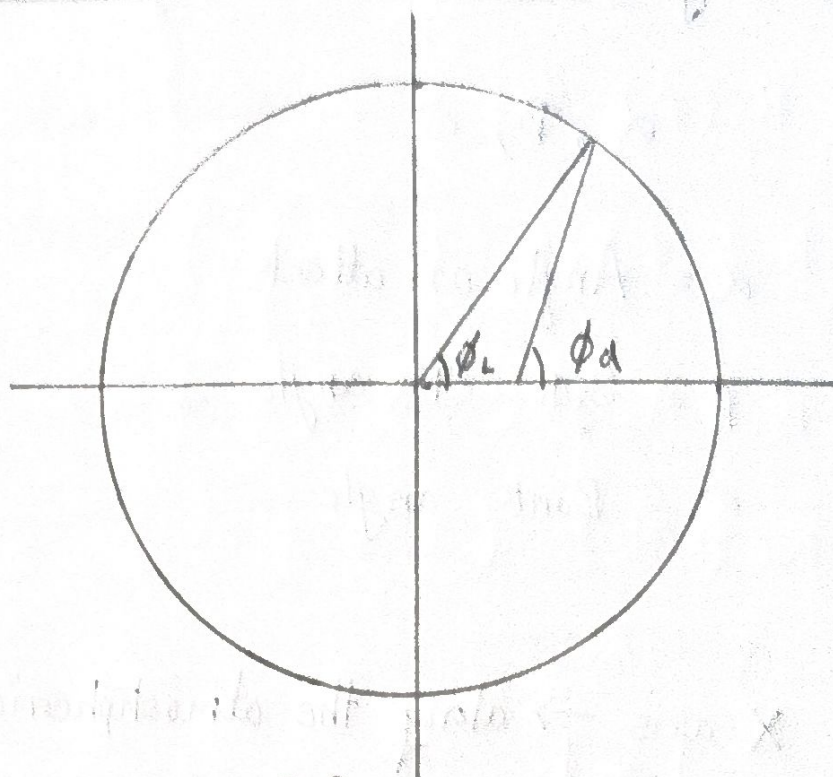
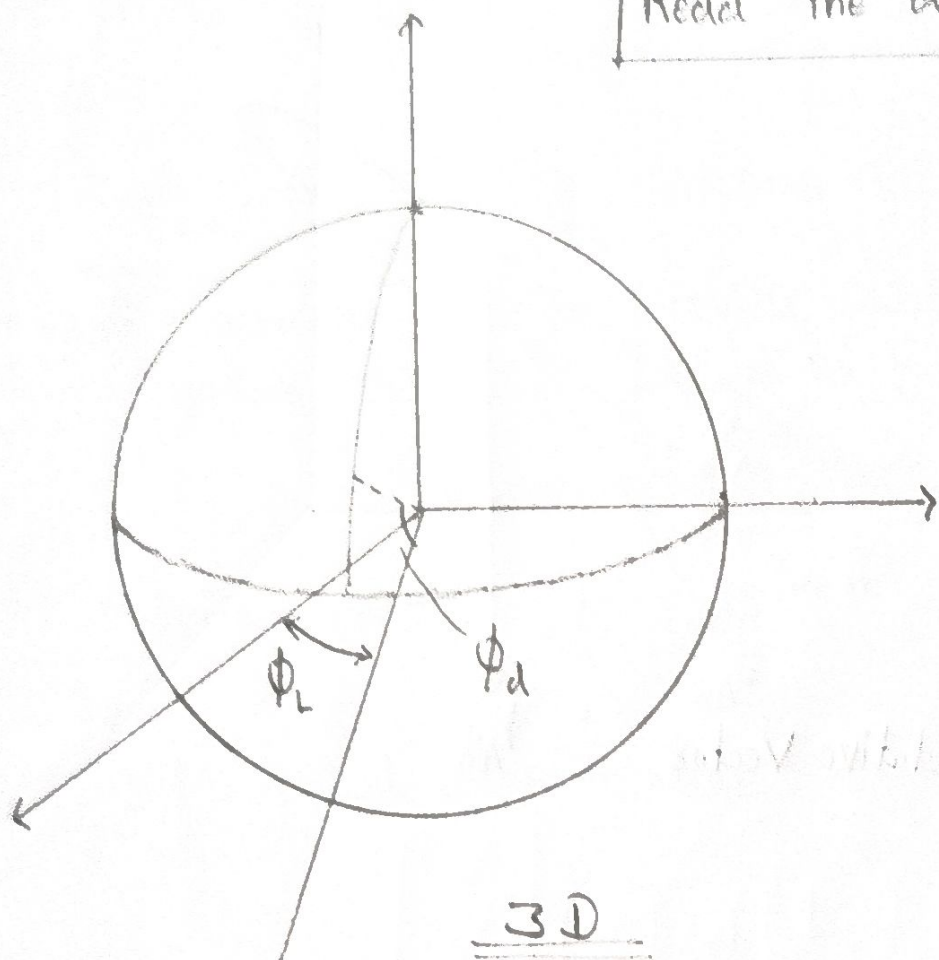
$\rightarrow$ Origin at ϕ_d, θ_L on the surface of earth.

X_L = along the A_{Z_L}

Z_L = points perpendicular to the earth surface inwards

Y_L = R.H.S triad

Read the documents for NASA post



2D

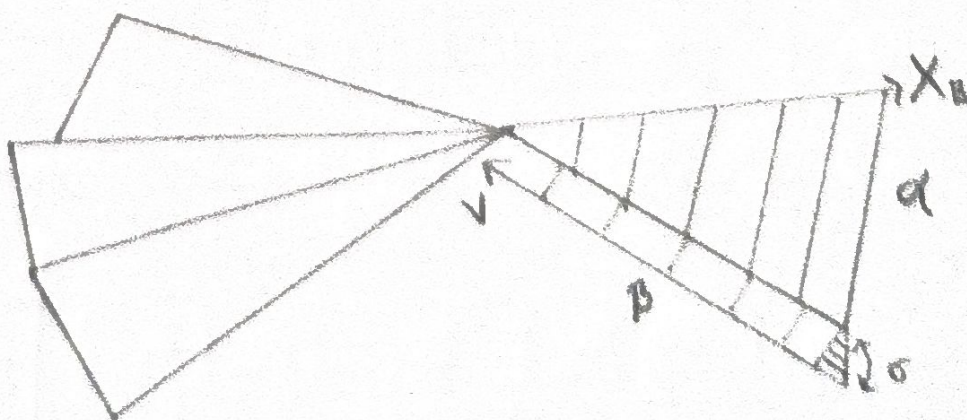
5 $\Rightarrow$ Body Frame

$X_B \rightarrow$ along vehicle's longitudinal axis

$Y_B \rightarrow$ towards right wing

$Z_B \rightarrow$ RHS triad

6 $\Rightarrow$ ARV frame (Atmospheric relative Velocity frame)



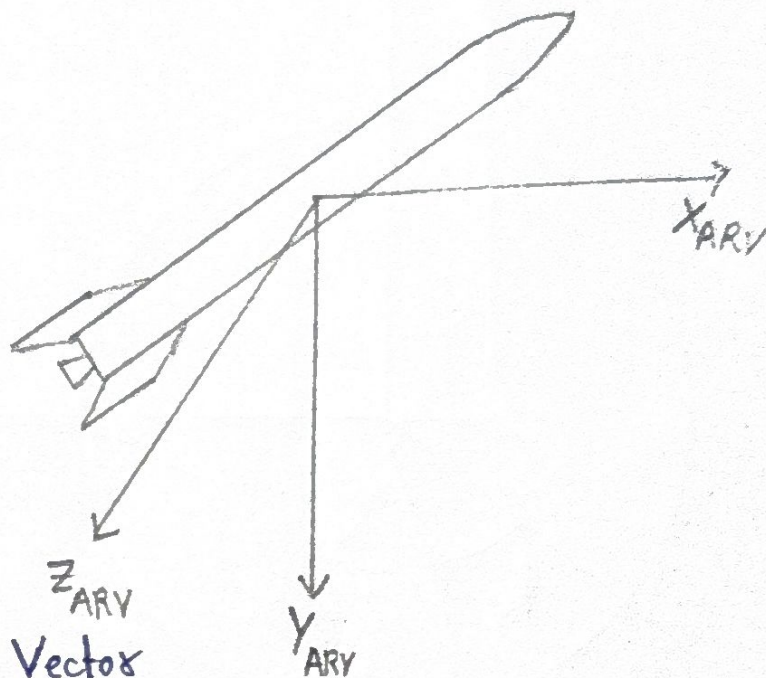
Aerodynamic Absolute Angle

α, β, σ

α = Angle of attack

β = Side Slip angle

σ = Bank angle



X-axis $\rightarrow$ along the atmospheric relative Vector

Y-axis $\rightarrow$ lies on horizon

Z-axis $\rightarrow$ RHS triad

Attitude Angles

Date: 06/02/2026

- Body frames exactly coincides with the Vehicle's orientation.
- Orientation of the Vehicle with respect to a given Coordinate System can be defined as:-

The orientation of the body frame, with respect to a given Coordinate System.

Eg:- Vehicle's orientation can be expressed w.r.t LPI frame.

↓
need a set of angles (ϕ_{pitch} , ψ_{yaw} , ϕ_{roll}) which on rotation about given axes of LPI frame will coincide with the body frame.

1 $\Rightarrow$ Inertial 'Euler Angle'

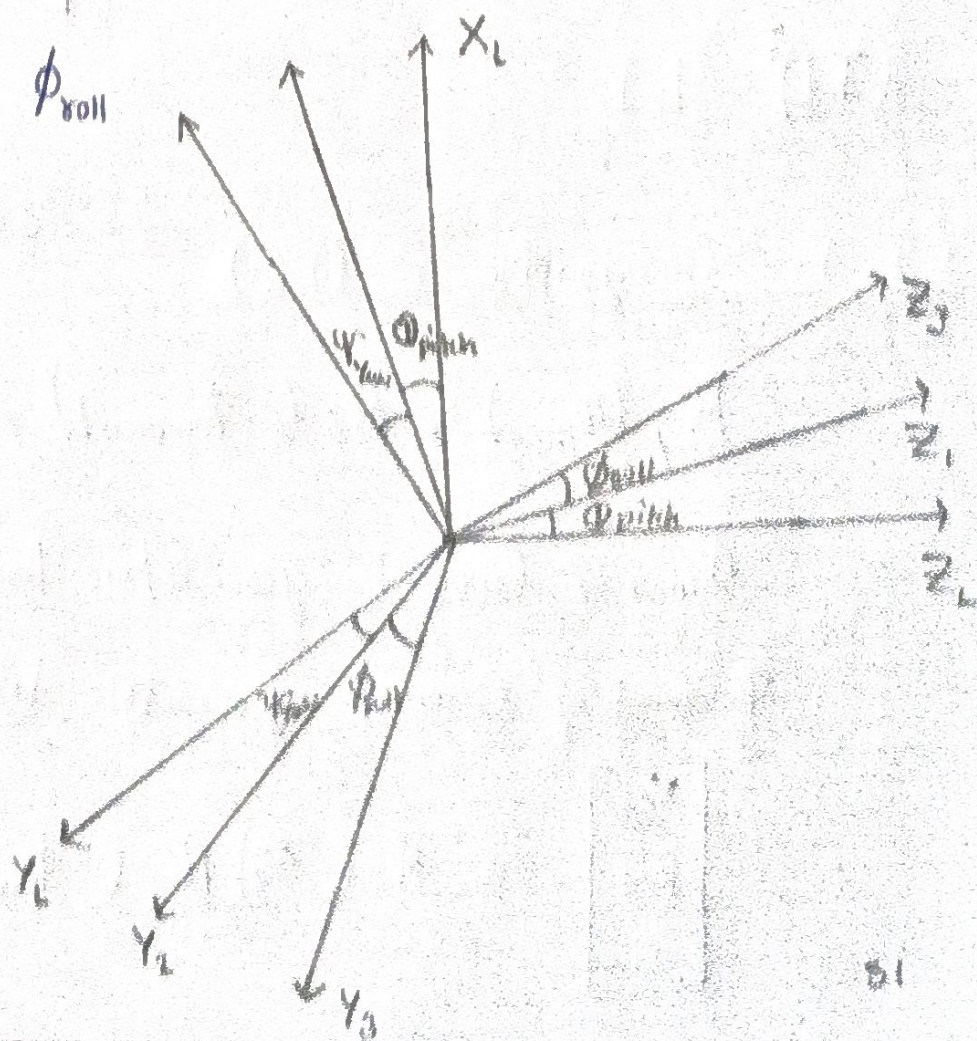
ϕ_{pitch} , ψ_{yaw} , ϕ_{roll}

2 $\Rightarrow$ Aerodynamic angles

α , β , σ

Rotation

1st - Pitch with respect to X axis



ECI to LPI frame :-

1 $\Rightarrow R_3(\phi_L)$ (about Z_1 axis)

2 $\Rightarrow R_2(-\phi_d)$ (about Y_2 axis)

3 $\Rightarrow R_1(-A_{z_L})$ (about X_2 axis)

4 $\Rightarrow R_2(-90^\circ)$ (about Y_3 axis)

$$\Rightarrow \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} = \underbrace{R_2(-90^\circ) R_1(-A_{z_L}) R_2(-\phi_d) R_3(\phi_L)}_{\text{expand}} \begin{bmatrix} X_I \\ Y_I \\ Z_I \end{bmatrix}$$

$$\begin{bmatrix} \cos \phi_L & \sin \phi_L & 0 \\ -\sin \phi_L & \cos \phi_L & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow (IL)^T = LI$$

ECI to Geographic :- (IG)

$$R_3(\phi_{iner}), R_2(-\phi_{current}), R_2(-\pi/2)$$

$\phi_{iner} \Rightarrow$ Current geocentric longitude

$\phi_{current} \Rightarrow$ Current geocentric latitude

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = R_2(-\pi/2) R_2(-\phi_{current}) R_3(\phi_{iner}) \begin{bmatrix} X_I \\ Y_I \\ Z_I \end{bmatrix}$$

Geographic to ARV frame :-

$$R_3(AZ_{rel})$$

$$R_2(V_{rel})$$

$$\begin{bmatrix} X_{ARV} \\ Y_{ARV} \\ Z_{ARV} \end{bmatrix} = R_2(V_{rel}) R_3(AZ_{rel}) \begin{bmatrix} X_{cr} \\ Y_{cr} \\ Z_{cr} \end{bmatrix}$$

ARV to Body frame :-

$$R_1(\sigma)$$

$$R_3(-\beta)$$

$$R_2(\alpha)$$

$$\begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = R_2(\alpha) R_3(-\beta) R_1(\sigma) \begin{bmatrix} X_{ARV} \\ Y_{ARV} \\ Z_{ARV} \end{bmatrix}$$

LPI to Body frame :-

$$R_2(\phi_{pitch})$$

$$R_3(\psi_{yaw})$$

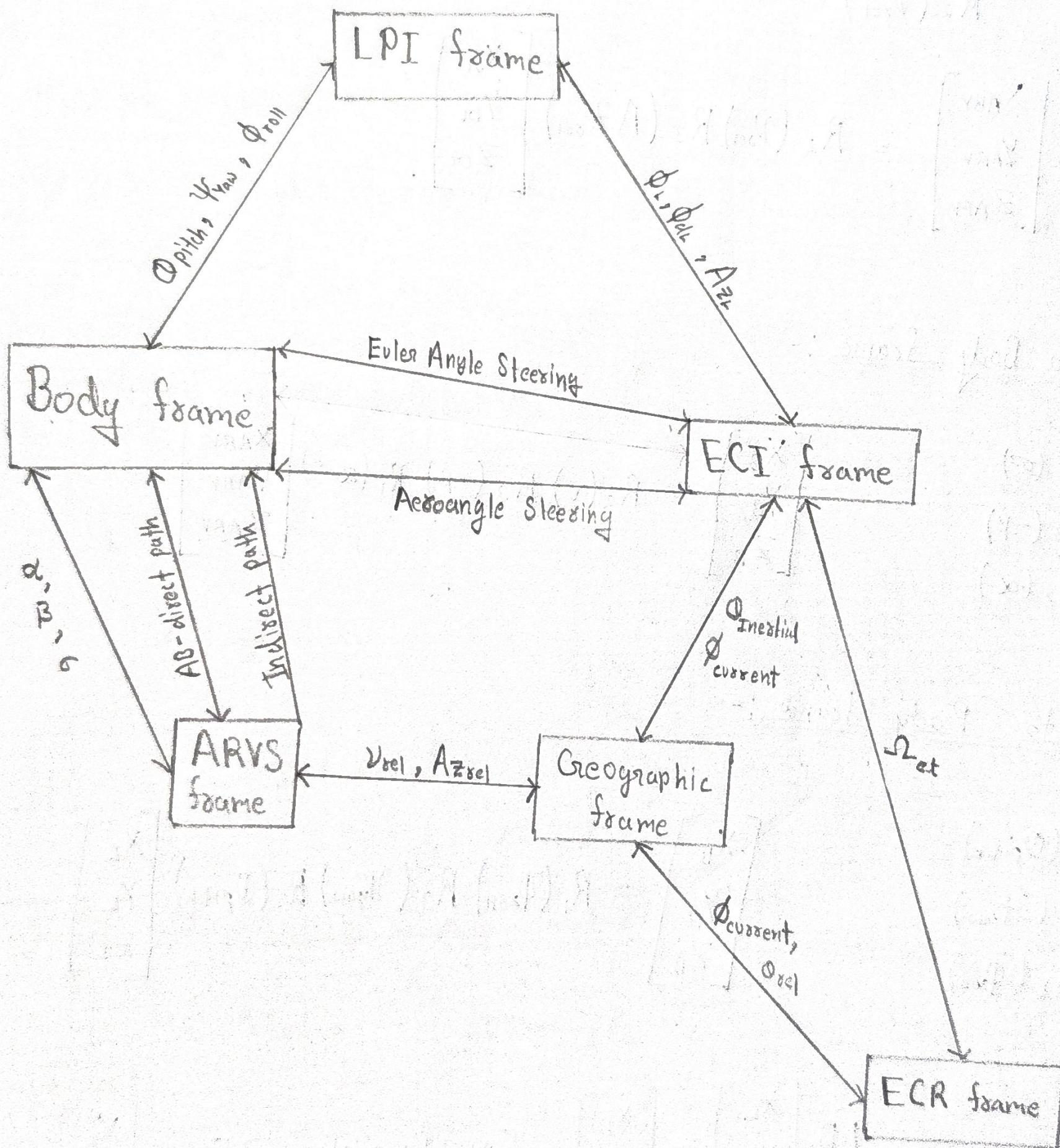
$$R_1(\phi_{roll})$$

$$\begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = R_1(\phi_{roll}) R_3(\psi_{yaw}) R_2(\phi_{pitch}) \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix}$$

$$LB \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix} = \begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = [AB] [GA] [IG] [LI] \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix}$$

$$LB = [AB] [GA] [IG] [LI]$$

Summary of Co-ordinate Transformation



$$\Rightarrow \Omega_{rel} = \Omega_{iner} - \Omega_{et}$$

Indirect path

(i) ARV $\rightarrow$ Geographic

$$\begin{bmatrix} X_G \\ Y_G \\ Z_G \end{bmatrix} = A_G \begin{bmatrix} X_{ARV} \\ Y_{ARV} \\ Z_{ARV} \end{bmatrix}$$

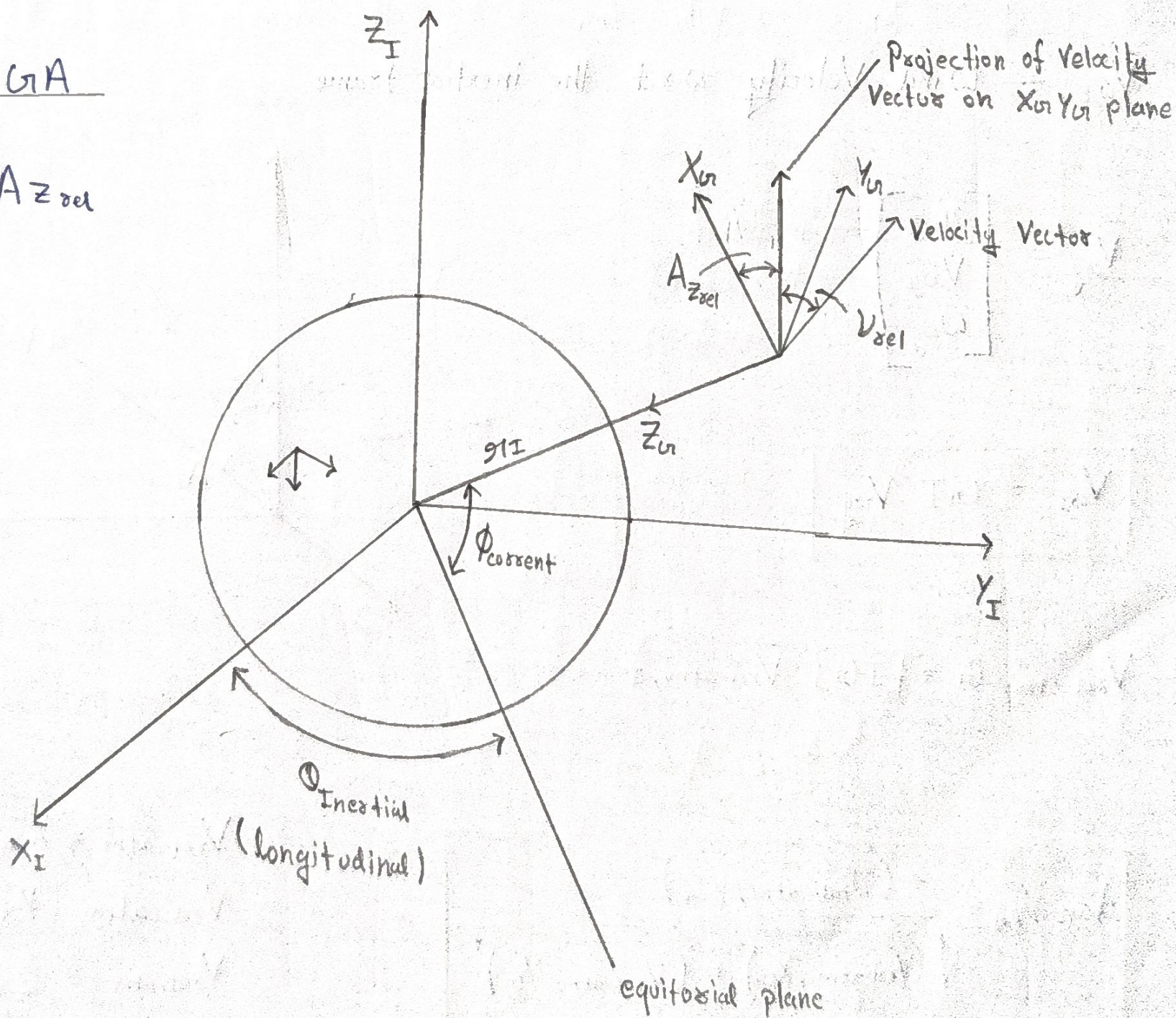
$[GA]$ is known to us

↓
Sequence

$$R_2(V_{rel}) R_3(AZ_{rel})$$

A_G or G_A

V_{rel} , AZ_{rel}



$$\Rightarrow \vec{r}_I = [r_{xI} \quad r_{yI} \quad r_{zI}]$$

$$\Rightarrow \vec{V}_{abs, I} = [V_{abs, xI} \quad V_{abs, yI} \quad V_{abs, zI}] \quad \{ \text{abs - absolute} \}$$

$$\Rightarrow \vec{V}_{rel, I} = \vec{V}_{abs, I} - \vec{\omega}_E \times \vec{r}_I$$

$$\Rightarrow \vec{V}_{rel-atm, I} = \vec{V}_{abs, I} - \vec{\omega}_E \times \vec{r}_I - \vec{V}_{wI}$$

We can neglect for Calculation and assignment.

$$\Rightarrow \vec{V}_{wI} \text{ --- Wind Velocity w.r.t the inertial frame}$$

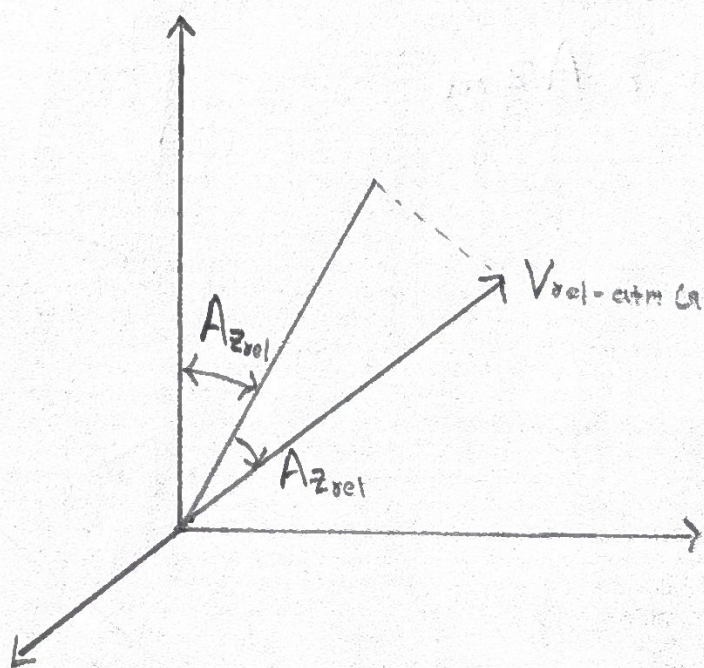
$$\Rightarrow \vec{V}_{wg} = \begin{bmatrix} u_{wg} \\ v_{wg} \\ w_{wg} \end{bmatrix} \begin{array}{l} \rightarrow \text{meridian} \\ \rightarrow \text{horizontal} \\ \rightarrow \text{vertical} \end{array}$$

$$\Rightarrow \vec{V}_{wI} = G_I \vec{V}_{wg}$$

$$\Rightarrow \vec{V}_{rel-atm, G} = [I \quad G] \vec{V}_{rel-atm, I}$$

$\phi_{iner}, \phi_{current}$

$$\Rightarrow \tan \gamma_{rel} = \frac{-(V_{rel-atm, zG})}{\sqrt{(V_{rel-atm, xG})^2 + (V_{rel-atm, yG})^2}}$$



$$V_{rel-atm} = x_G$$

$$V_{rel-atm} = y_G$$

$$V_{rel-atm} = z_G$$

$$\Rightarrow \sin \phi_{\text{current}} = \frac{\eta_{zI}}{|\eta_I|}$$

$$\Rightarrow \tan \phi_{\text{iner}} = \frac{\eta_{yI}}{\eta_{xI}}$$

$$\Rightarrow LB = [LB] [IL] [GI] [AG] \begin{bmatrix} X_{ARV} \\ Y_{ARV} \\ Z_{ARV} \end{bmatrix}$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 ϕ_{pitch} ϕ_L ϕ_{iner} V_{rel}
 ψ_{yaw} ϕ_d ϕ_{current} $A_{z, \text{rel}}$
 ϕ_{roll} A_{zL}

$$\Rightarrow \begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = AB \begin{bmatrix} X_{ARV} \\ Y_{ARV} \\ Z_{ARV} \end{bmatrix}$$

Indirect path $\rightarrow$ $AB = \begin{bmatrix} AB_{11} & AB_{12} & AB_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$\tan \alpha = \frac{A_{31}}{AB_{11}}$

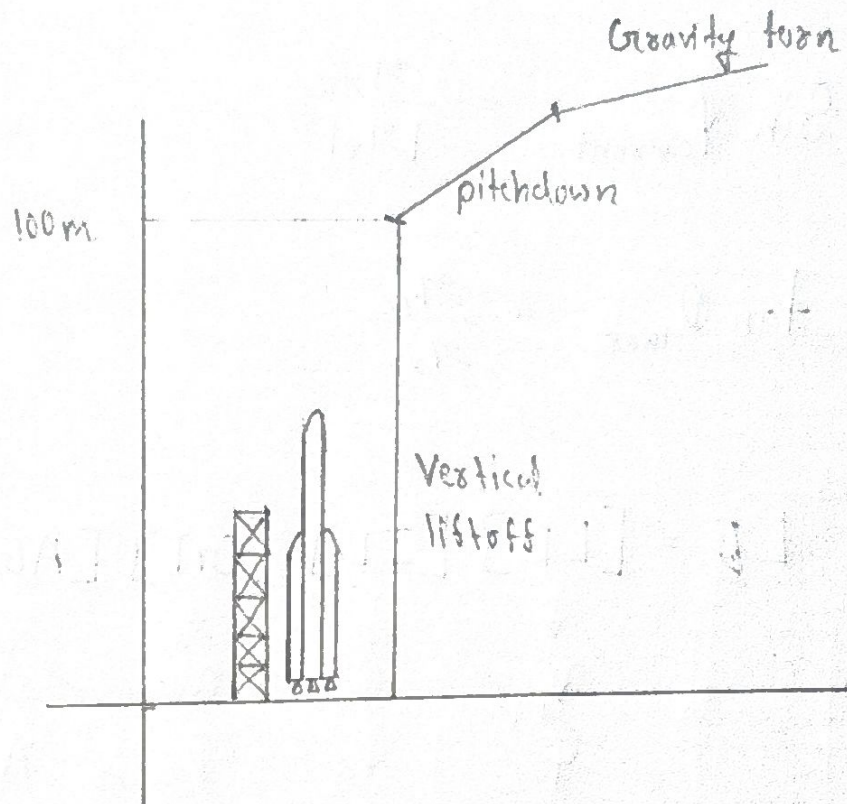
$$\Rightarrow \beta = \sin^{-1}(A_{21})$$

$$\Rightarrow \tan(\sigma) = \frac{A_{23}}{A_{22}}$$

Direct path — LHS $\rightarrow AB \rightarrow \alpha, \beta, \sigma$

$$\Rightarrow \begin{bmatrix} \cos \alpha \cos \beta & -\cos \alpha \sin \beta \cos \sigma + \sin \alpha \sin \sigma & -\cos \alpha \sin \beta \sin \sigma - \sin \alpha \cos \sigma \\ \sin \beta & \cos \beta \cos \sigma & \cos \beta \sin \sigma \\ \sin \alpha \cos \beta & -\sin \alpha \sin \beta \cos \sigma - \cos \alpha \sin \sigma & -\sin \alpha \sin \beta \sin \sigma + \cos \alpha \cos \sigma \end{bmatrix}$$

Euler Angle Steering



Initial Angle of flight :-

- Vertical lift off
- Constant pitch rate (pitch down)

$\dot{\theta}, \dot{\psi}, \dot{\phi} \rightarrow$ Known or Values imposed by the design based on the ~~envi~~ req.

determine $\alpha, \beta, \sigma \rightarrow \} C_A, C_N, C_S$
multivariable

$$\begin{bmatrix} \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} C_A \\ C_N \\ C_S \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \sigma \end{bmatrix}$$

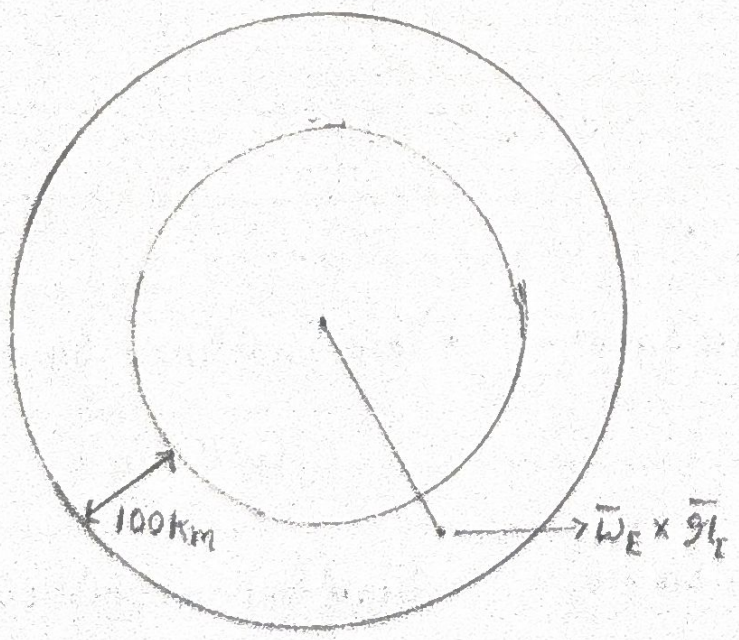
Note :-

$$\begin{bmatrix} V_{absolute} \\ A_{Z_{absolute}} \end{bmatrix} = \begin{bmatrix} V_{rel-at m Gr} \\ A_{Z_{rel}} \end{bmatrix}$$

$$V_{abs, I} \Rightarrow V_{abs, Gr}$$

ARV $\rightarrow Gr \in \in$

$B \leftarrow L \leftarrow I \leftarrow$



After 100 km there is no atmosphere
So, need not to Calculate $\bar{W}_E \times \bar{g}_I$

Aerodynamic Angle Steering

Aerodynamic angles are set to be zero to minimize the ~~error~~ aerodynamic load on the Vehicle

$$\alpha = 0, \beta = 0, \sigma = 0$$

need to provide $\dot{\theta}, \dot{\psi}, \dot{\phi}$ such that during gravity turn $\alpha = 0, \beta = 0, \sigma = 0$

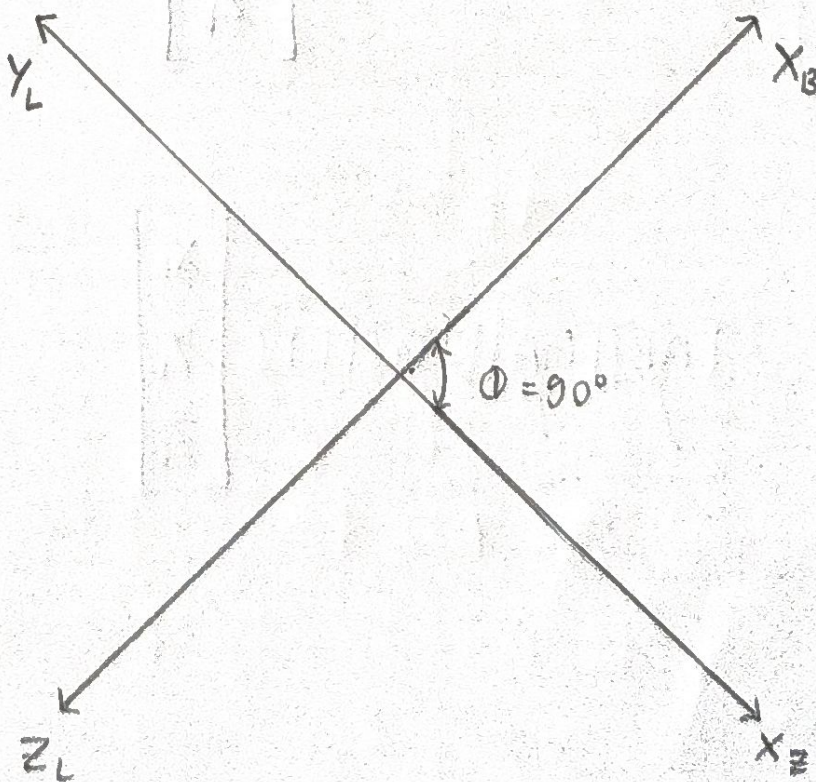
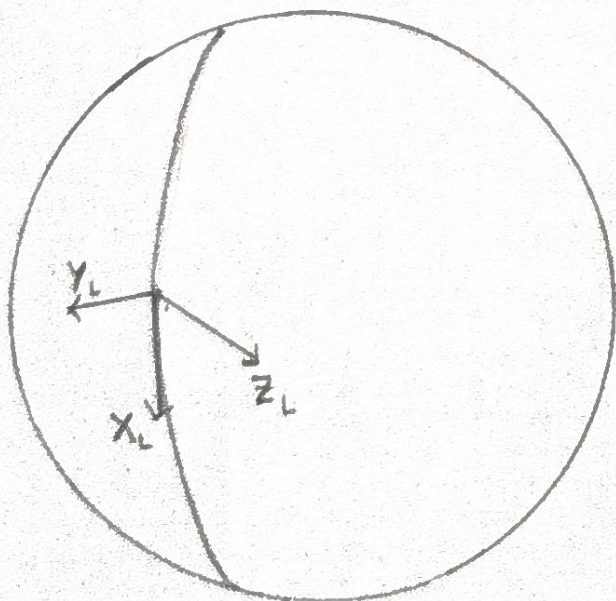
ΔT

$$t_0 \rightarrow t_0 + \Delta t$$

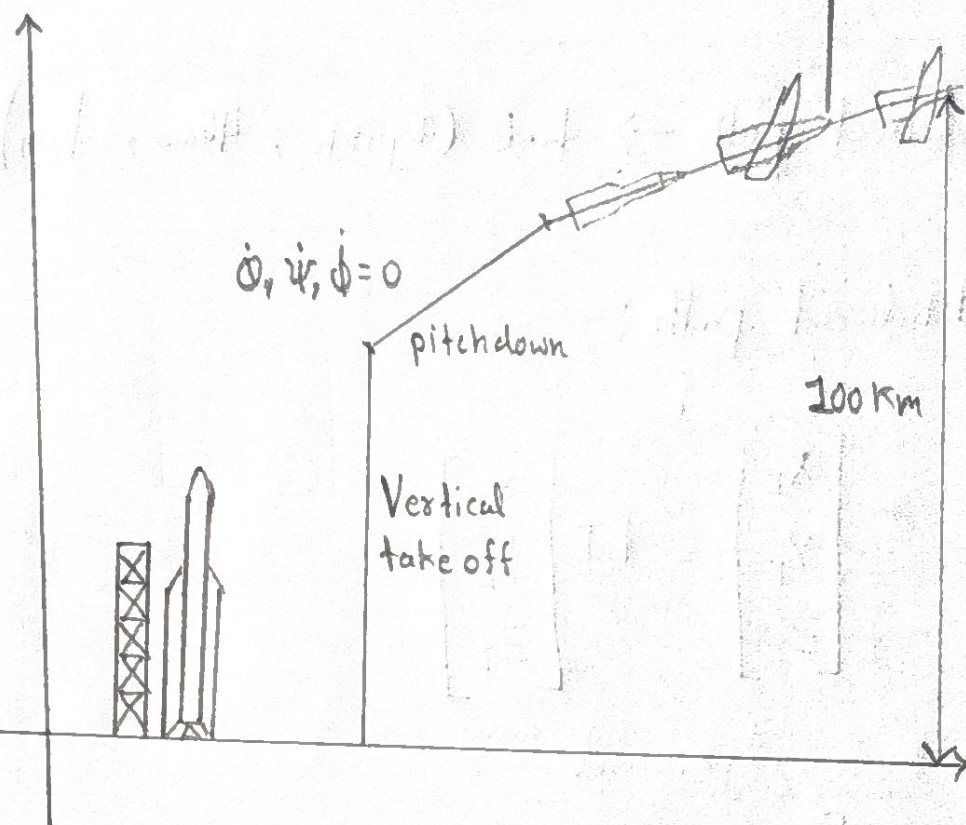
$$\theta_0 \rightarrow \theta$$

$$\psi_0 \rightarrow \psi$$

$$\phi_0 \rightarrow \phi$$



aerodynamic load To minimize $\alpha = 0, \beta = 0, \sigma = 0$



impose $\alpha = 0, \beta = 0, \sigma = 0$

$$\dot{\theta} = \frac{\theta - \theta_0}{\Delta t}, \quad \dot{\phi} = \frac{\phi - \phi_0}{\Delta t}$$

$$\dot{\psi} = \frac{\psi - \psi_0}{\Delta t}$$

How to find out ϕ_{pitch} , ψ_{yaw} , ϕ_{roll}

Direct path $\rightarrow$ L.B (ϕ_{pitch} , ψ_{yaw} , ϕ_{roll})

Indirect path:-

$$(i) \begin{bmatrix} x_I \\ y_I \\ z_I \end{bmatrix} = LI \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix}$$

$$(ii) \begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix} = [IG][LI] \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix}$$

$$(iii) \begin{bmatrix} x_{ARV} \\ y_{ARV} \\ z_{ARV} \end{bmatrix} = [GA][IG][LI] \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix}$$

$$(iv) \begin{bmatrix} x_B \\ y_B \\ z_B \end{bmatrix} = \overbrace{[AB][GA][IG][LI]}^{LB} \begin{bmatrix} x_L \\ y_L \\ z_L \end{bmatrix}$$

$\begin{matrix} \swarrow & \swarrow & \swarrow & \swarrow \\ d=0 & \psi_{rel} & \phi_{inco} & \phi_d \\ \beta=0 & A_{rel} & \phi_{current} & \phi_L \\ \sigma=0 & & & \end{matrix}$

LB direct path :-

LB indirect path :-

$$\begin{bmatrix} \cos \psi_{yaw} \cos \phi_{pitch} & \sin \psi_{yaw} & -\cos \psi_{yaw} \sin \phi_{pitch} \\ (-) & \cos \phi_{roll} \cos \psi_{yaw} & (-) \\ (-) & -\sin \phi_{roll} \cos \psi_{yaw} & (-) \end{bmatrix} \Leftrightarrow \begin{bmatrix} LB_{11} & LB_{12} & LB_{13} \\ LB_{21} & LB_{22} & LB_{23} \\ LB_{31} & LB_{32} & LB_{33} \end{bmatrix}$$

$$\sin \psi_{yaw} = LB_{21}$$

find
 ψ_{yaw}

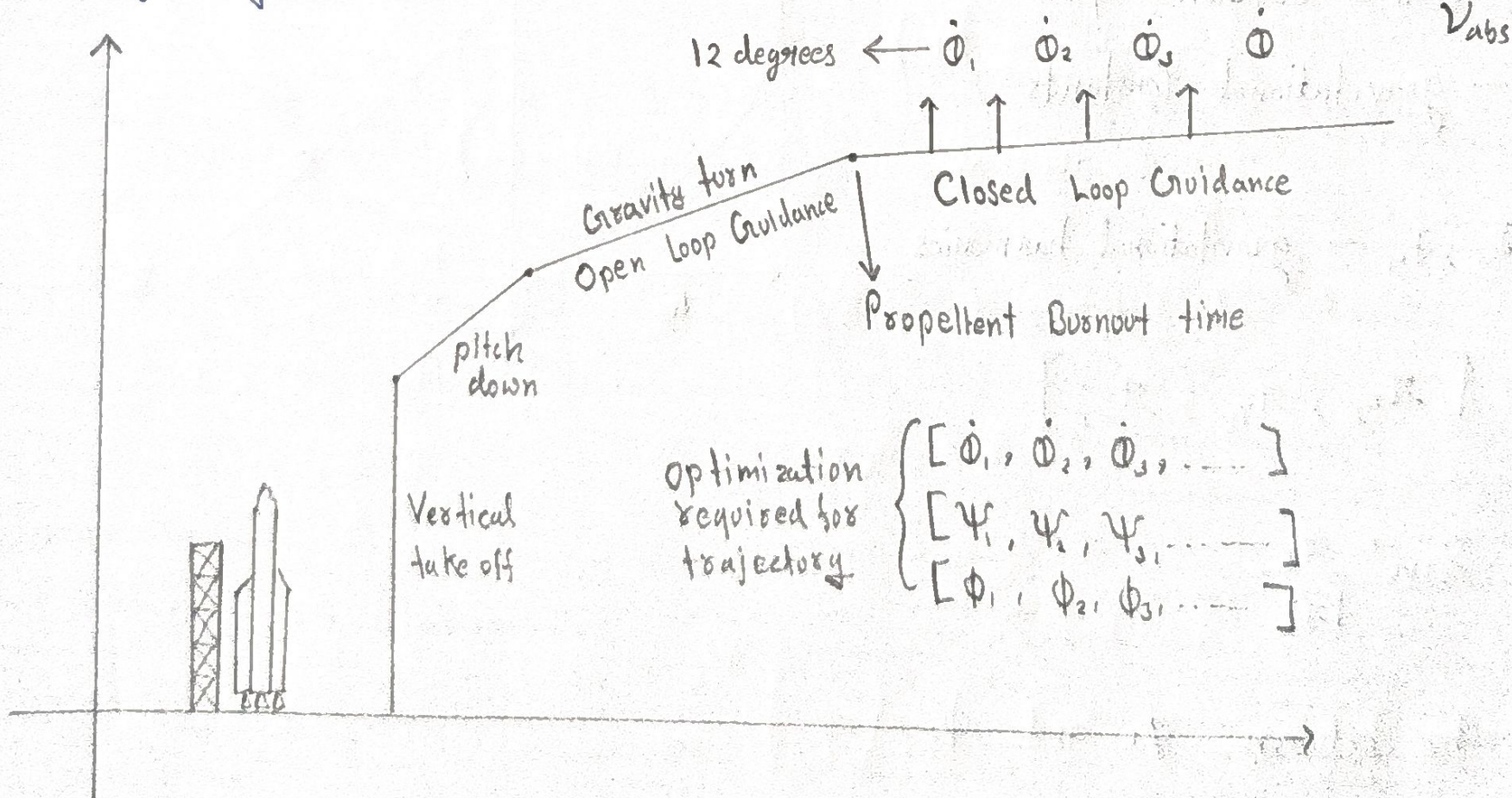
$$\tan \phi_{roll} = \frac{-LB_{3,2}}{LB_{2,2}}$$

ϕ_{roll}

$$\tan \phi_{pitch} = \frac{-LB_{1,3}}{LB_{1,1}}$$

ϕ_{pitch}

→ after gravity turn overs (open loop guidance over), Closed loop guidance,



We will Cutoff the engine for some time to slow down the Velocity then start again.

Planet Model

Date :- 17/02/2026

Information about the assignment :- Run it as a Simulator we are to go to some location (try to go),

Go ahead with your own sizing of rocket
(Playing with numbers) (Combination of different engines)

+ trajectory

+ Don't forget to counter the moment.

(i) Oblate Spheroid

R_e - equatorial radius

R_p - polar radius

ω_E - earth's rotational speed

μ_E - gravitational constants

J_2, J_3, J_4 - gravitational harmonics

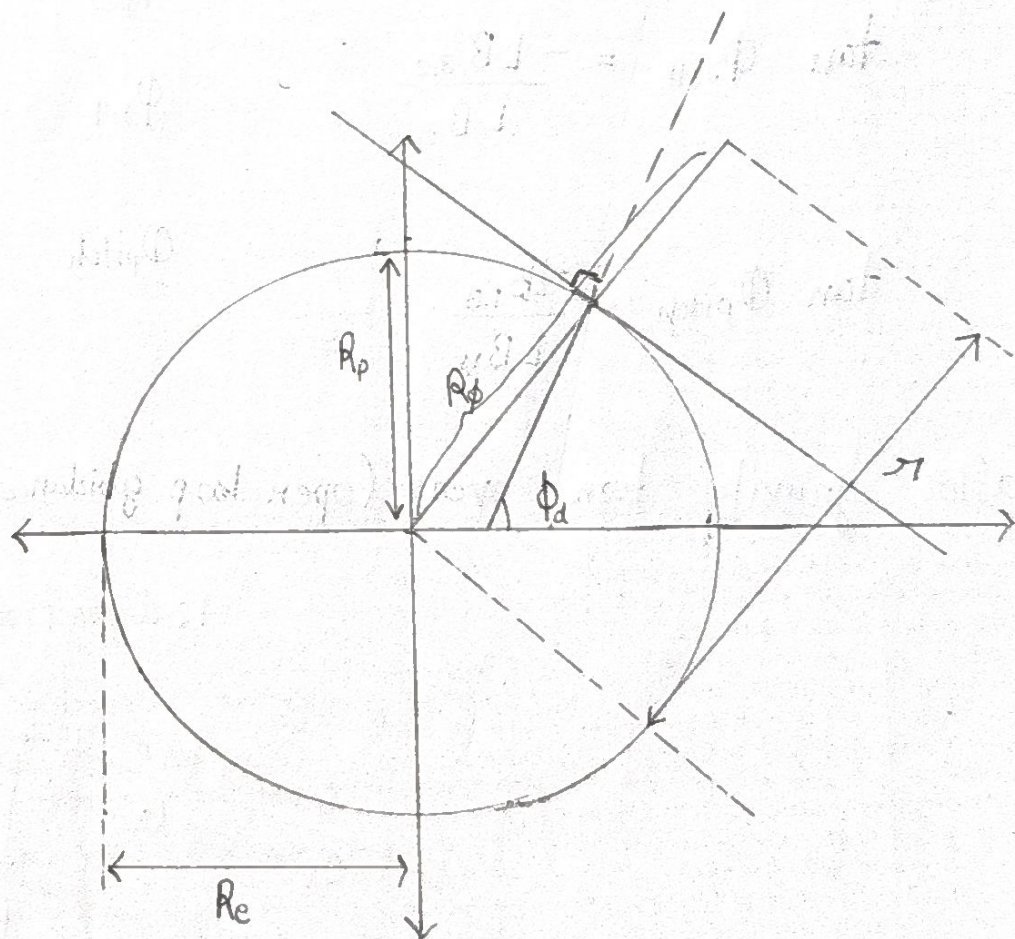
$$\vec{g} = [g_{x_I}, g_{y_I}, g_{z_I}]$$

$$\phi = \sin^{-1} \frac{g_x}{|\vec{g}|}$$

Relationship between ϕ_d and ϕ

$$(1-f) \tan \phi_d = \tan \phi$$

$$f = \frac{R_e - R_p}{R_e}$$



$$\phi_d = \tan^{-1}(k \tan \phi), \quad k = \left(\frac{R_e}{R_p} \right)^2$$

$$R_\phi = \frac{R_e}{\sqrt{1 + (k-1)\sin^2 \phi}}$$

$$h = |\vec{r}| - R_\phi$$

(ii) Planet Gravitational model

$$\mu_E = 3.98600448 \times 10^{14} \text{ m}^3/\text{s}^2$$

$$J_2 = 1.0826368 \times 10^{-3}, \quad J_3 = 0, \quad J_4 = 0$$

$$G_{X_I} = -\mu_E \left(\frac{x_I}{|\vec{r}|^3} \right) \left[1 + 1.5 J_2 \left(\frac{R_e}{|\vec{r}|} \right)^2 \left(1 - 5 \left(\frac{z}{|\vec{r}|} \right)^2 \right) \right]$$

$$G_{Y_I} = -\mu_E \left(\frac{y_I}{|\vec{r}|^3} \right) \left[1 + 1.5 J_2 \left(\frac{R_e}{|\vec{r}|} \right)^2 \left(1 - 5 \left(\frac{z}{|\vec{r}|} \right)^2 \right) \right]$$

$$G_{Z_I} = -\mu_E \left(\frac{z_I}{|\vec{r}|^3} \right) \left[1 + 1.5 J_2 \left(\frac{R_e}{|\vec{r}|} \right)^2 \left(1 - 5 \left(\frac{z}{|\vec{r}|} \right)^2 \right) \right]$$

Some data

$$R_e = 6378137 \text{ m}$$

$$R_p = 6356752 \text{ m}$$

$$\mu_E = 3.98600448 \times 10^{14} \text{ m}^3/\text{s}^2$$

$$J_2 = 1.0826368 \times 10^{-3}$$

$$J_3 = 0$$

$$J_4 = 0$$

(iii) Planet Atmospheric Model

Indian Standard Atmosphere,

$$P, \rho, T \rightarrow f(h) \quad \xrightarrow{\text{Altitude}}$$

Speed of Sound $C_s = \sqrt{\gamma R T}$

$$C_s = 20.046803\sqrt{T}, \quad M = \frac{|\vec{V}_{rel, atm}|}{C_s}$$

International Standard Atmosphere,

- Troposphere (0-11 km)

-6.5°C/km lapse rate

- Stratosphere (11-25 km)

isothermal (-56.5°C or 216.65 K)

- Upper Stratosphere (>25 km)

+ 3.0°C/km inversion

Governing equation -

$$P = P_0 \left(\frac{T}{T_0} \right)^{-\frac{g}{LR}} \quad \begin{matrix} \text{LR - gas constant} \\ \text{lapse rate} \end{matrix}$$

$$P = P_0 \exp \left(\frac{-g(h-h_0)}{RT} \right)$$

Reference Sea level

$$T_0 = 288.15 \text{ K}$$

$$P_0 = 10325 \text{ Pa}$$

$$\rho = 1.225 \text{ kg/m}^3$$

$$\gamma = 1.4$$

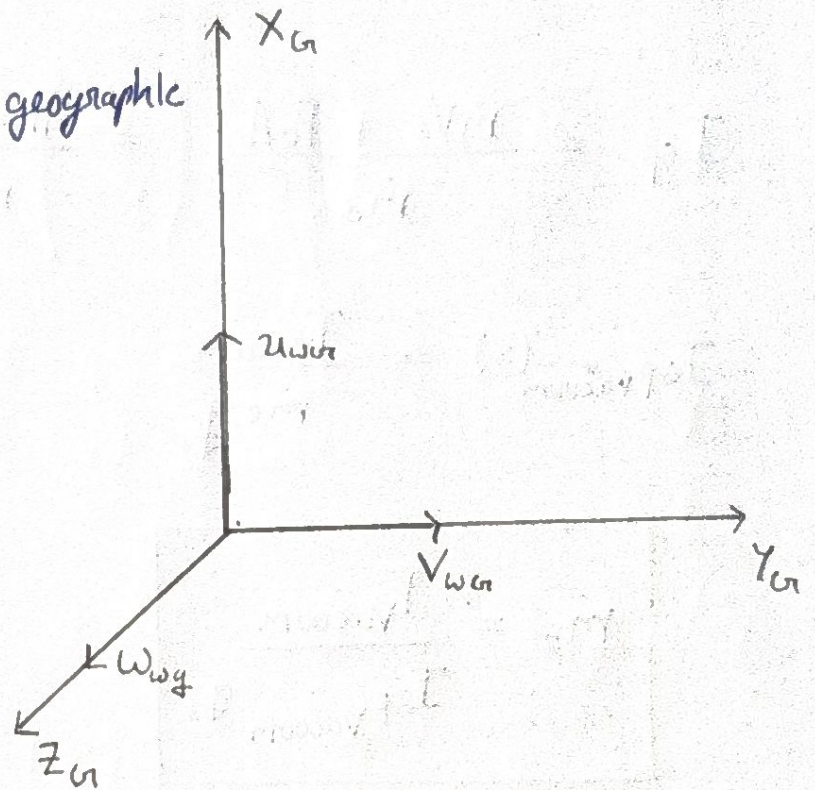
Planet Model

Date :- 18/02/2026

(iv) Wind Model

Wind Velocity data $\rightarrow$ tabulated with respect to geographic (G) frame

$$V_{wg} = \begin{bmatrix} u_{wg} \\ v_{wg} \\ w_{wg} \end{bmatrix} \begin{array}{l} \rightarrow \text{meridian} \\ \rightarrow \text{Zonal} \\ \rightarrow \text{Vertical} \end{array}$$



(v) Vehicle Model

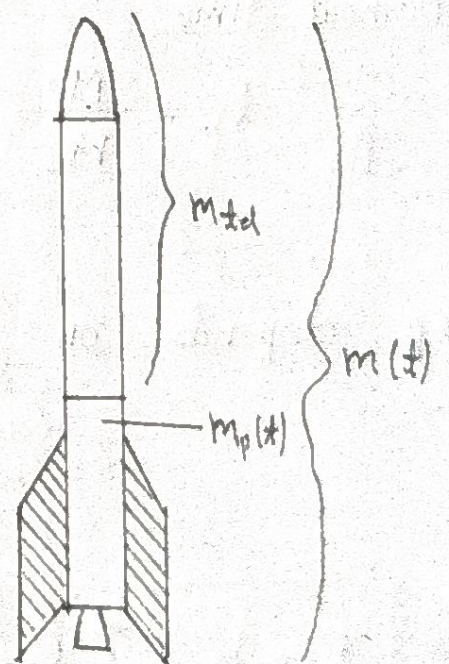
(a) Mass Properties Model

$$m(t) = m_{td}(t) + m_{p, \text{stage}}(t)$$

Instantaneous mass of the Vehicle

mass of the payload

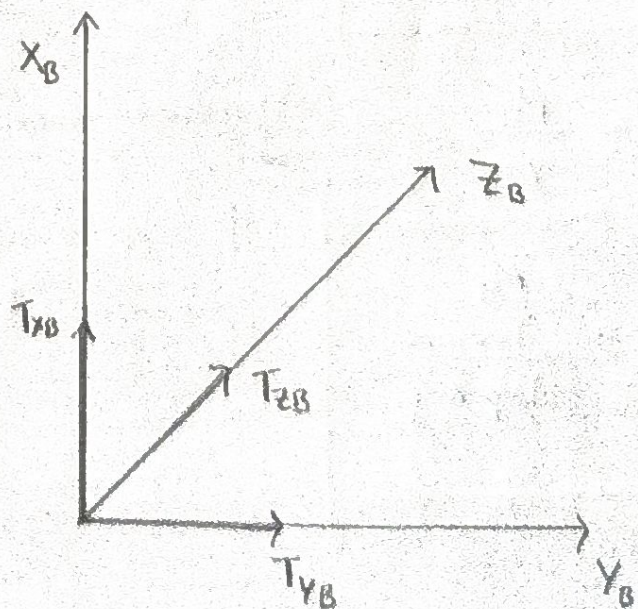
Instantaneous mass of the propulsion stage



(b) Propulsion Model :-

Thrust Coincides with body frame

$$\vec{f}_{TB} = \begin{bmatrix} T_{xB} \\ T_{yB} \\ T_{zB} \end{bmatrix} = \begin{bmatrix} T_{xB} \\ 0 \\ 0 \end{bmatrix} \quad \text{--- typically thrust velocity will introduce components}$$



$$\Rightarrow T_{net} = T_{XB} = T_{vacuum} - P_{atm} \cdot A_e$$

$$\Rightarrow = \dot{m} V_e + P_e A_e - P_{atm} \cdot A_e$$

$$\Rightarrow I_{sp} = \frac{\dot{m} V_e + P_e A_e}{\dot{m}_e} = \frac{T_{vacuum}}{\dot{m}_e}$$

$$\Rightarrow I_{sp_{vacuum}} (s) = \frac{T_{vacuum}}{\dot{m}_e g_0}$$

$$\Rightarrow \boxed{\dot{m}_e = \frac{T_{vacuum}}{I_{sp_{vacuum}} g_0}}$$

Burnout time

$$\Rightarrow t_b = \frac{M_p}{\dot{m}_e}$$

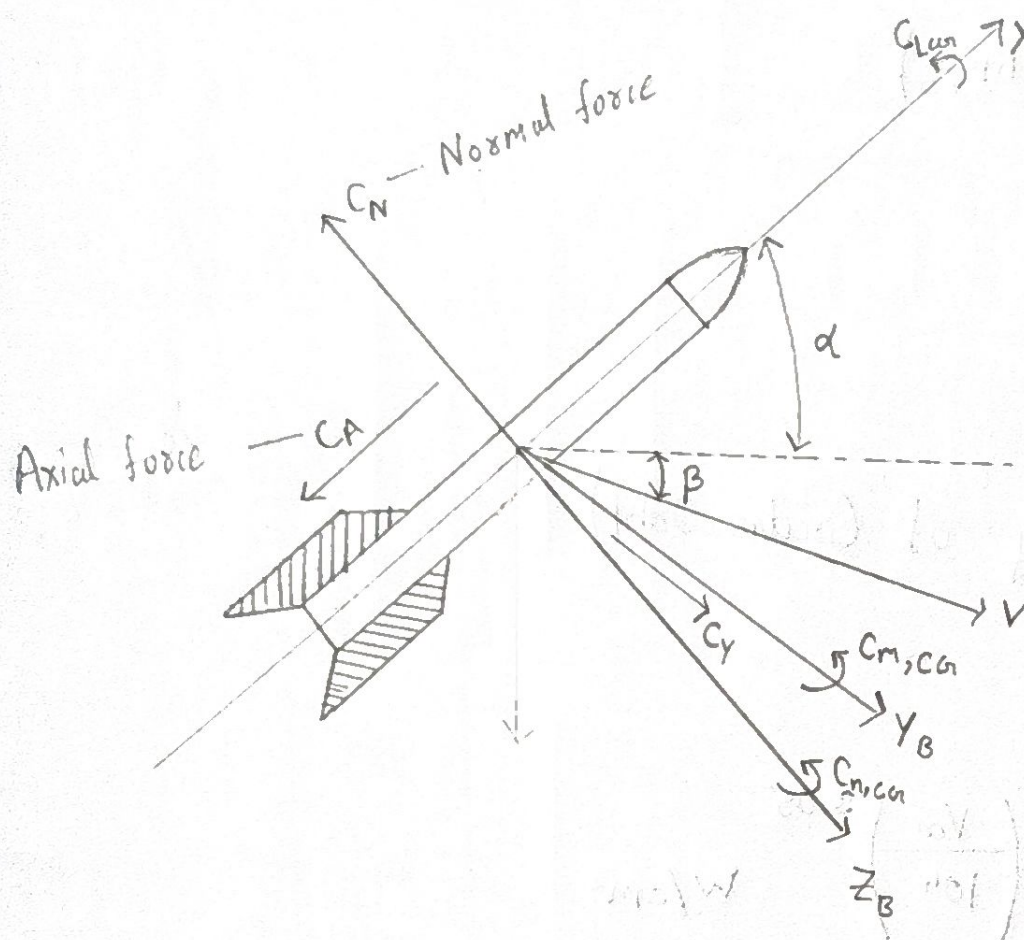
I_{sp} depends on $\underbrace{C^*}_{\text{Combustion efficiency}}$ & $\underbrace{C_F}_{\text{nozzle dependent}}$

$$\Rightarrow \boxed{I_{sp} = C^* \cdot C_F}$$

$$\Rightarrow t_b = \frac{M_p}{\dot{m}_e}$$

$$\Rightarrow \boxed{t_b = \frac{M_p}{I_{sp} \times g_0}}$$

(c) Aerodynamic Model



- Missiles will have high Structural Strength to resist it.
- In Sounding Rocket: location of C_p should be below C.G.
- Data that we use going to get from scaling model wind tunnel testing

$$\vec{F}_{AB} = qS \begin{bmatrix} -C_A \\ C_Y \\ -C_N \end{bmatrix} = qS \begin{bmatrix} -(C_{A, \text{basic}} + \Delta C_A) \\ (C_{Y, \text{basic}} + \Delta C_Y) \\ -(C_{Z, \text{basic}} + \Delta C_Z) \end{bmatrix}$$

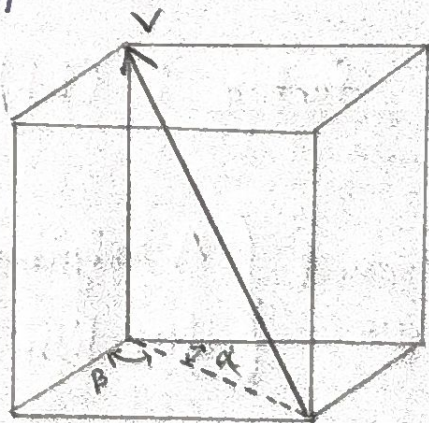
$C_A, C_N, C_Y \rightarrow$ function of (α) and (β) for each mach number (M)

$\Delta C_A, \Delta C_N, \Delta C_Y \rightarrow$ incremental area coefficient due to fin deflection

C_D vs M for α, β , or α_{total} $M = \frac{|V_{\text{vel}} - a_{\text{tm}, p}|}{C_s}$

$$\alpha_{\text{total}} = \vec{\beta} + \vec{\alpha}$$

Basically α Resultant,



Aerodheating Calculations

Date:- 19/02/2026

heat - flux ($|V_{\text{rel-atm}}| < 2.5 \text{ km/s}$)

$$\Rightarrow Q = \frac{11030}{\sqrt{R_n}} \left(\frac{\rho_{\infty}}{\rho_{sl}} \right) \left(\frac{V_{\infty}}{V_{c_0}} \right)^{3.15}$$

Sea level

$$\Rightarrow V_{c_0} = -7.950 \text{ km/s} \quad (\text{Velocity of Circular Orbit})$$

Velocity $> 2.5 \text{ km/s}$

$$\Rightarrow Q = \frac{18300}{\sqrt{R_n}} \cdot (\rho_{\infty})^{0.5} \left(\frac{V_{\infty}}{10^4} \right)^{3.05} \text{ W/cm}^2$$

R_n - nose radius (m)

ρ_{∞} - free stream density (kg/m^3)

ρ_{sl} - Sea level density (kg/m^3)

dynamic pressure

$$\Rightarrow q = \frac{1}{2} \rho_{\infty} V_{\infty}^2, \quad V_{\infty} \rightarrow [V_{\text{rel-atm}}]$$

Load factor :- (Lf)

expressed in g's $\left\{ \begin{array}{l} \text{Aerodynamic force acting perpendicular to vehicle's longitudinal axis} \\ \text{direction opposite to } Z_p \end{array} \right.$

$$\Rightarrow Lf = \frac{-\int_{AB} Z_B}{9.81} = \frac{q S C_N}{9.81}$$

RAS Aero

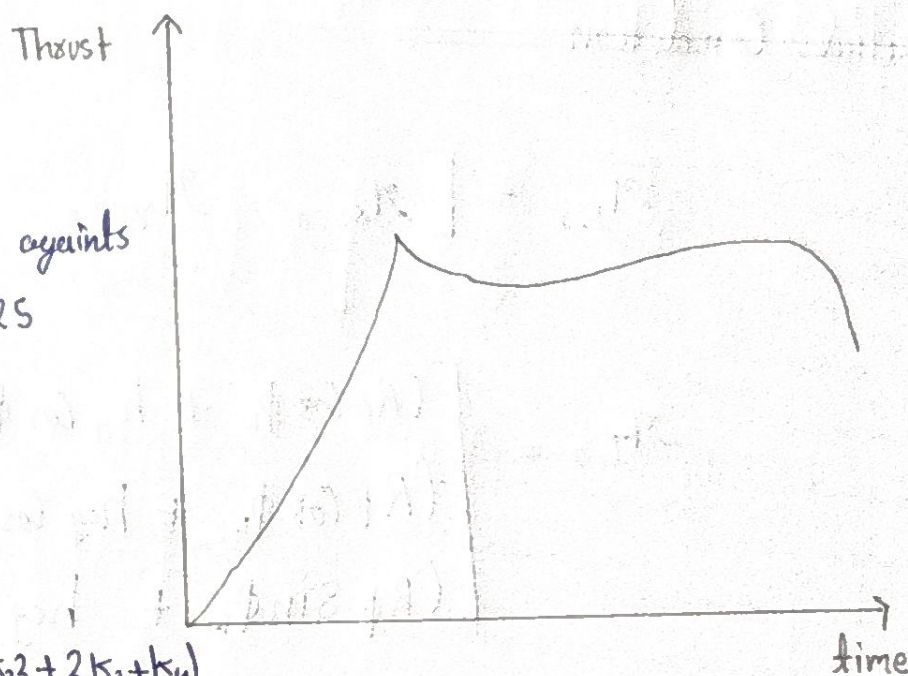
$$\alpha = 0^\circ, 2^\circ, 4^\circ \rightarrow C_D, C_N, X_{EA},$$

$C_a \rightarrow$ data against
 $0 < M < 25$

Rk4 Method

$$\Rightarrow X_N = \begin{bmatrix} \eta_{x_I} \\ \eta_{y_I} \\ \eta_{z_I} \\ V_{x_I} \\ V_{y_I} \\ V_{z_I} \end{bmatrix}$$

$$X_{N+1} = X_N + \frac{\Delta t}{6} (k_1 + k_2 + 2k_3 + k_4)$$



Thrust vs Time for Star grain
Solid motor

$$\Rightarrow m(t) \cdot \frac{d\vec{V}_I}{dt} = \vec{f}_{AI} + \vec{F}_{T,I} + m(t) \vec{G}_I$$

$\downarrow$
 $\vec{F}_{AB} \quad \text{(or)} \quad F_{A,ARV}$

Aerodynamic force in body frame.

$$\Rightarrow \frac{d\vec{V}_I}{dt} = \frac{1}{m(t)} \left[\underbrace{[BI]}_{\substack{\text{if } [AI] \\ \text{we use } \vec{F}_{A,ARV}}} \vec{F}_{AB} + \underbrace{[BI]}_{\substack{\text{Body to inertial} \\ \text{or} \\ [IB]^{-1} \text{ inertial to body}}} \vec{f}_{T,B} \right] + \vec{G}_I$$

if $[AI]$
we use $\vec{F}_{A,ARV}$

Body to inertial

or
 $[IB]^{-1}$ inertial to body

Inertial to LPI, LPI to body
When launched $\text{Pitch} = 90^\circ$

Initial Condition

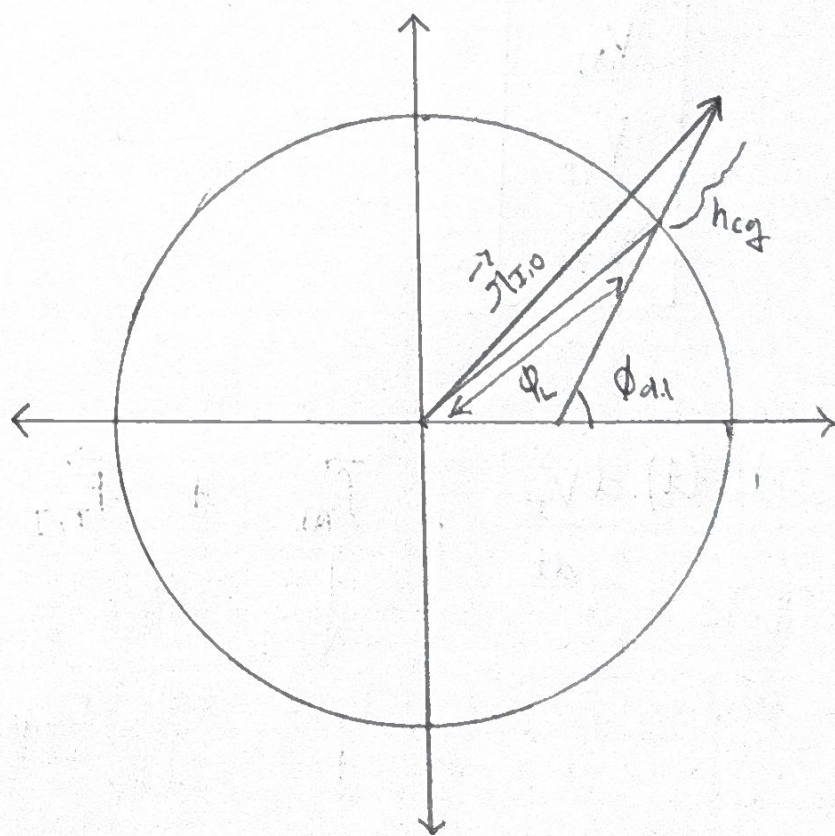
$$\Rightarrow \vec{r}_{I,0} = \begin{bmatrix} r_{xI,0} & r_{yI,0} & r_{zI,0} \end{bmatrix}$$

$$\Rightarrow r_{I,0} = \begin{bmatrix} (R\phi \cos \phi_L + h_{eq} \cos \phi_{dL}) \cos \theta_L \\ (R\phi \cos \phi_L + h_{eq} \cos \phi_{dL}) \sin \theta_L \\ (R\phi \sin \phi_L + h_{eq} \sin \phi_{dL}) \end{bmatrix} = \begin{bmatrix} r_{xI0} \\ r_{yI0} \\ r_{zI0} \end{bmatrix}$$

$$\Rightarrow V_{I,0} = \begin{bmatrix} V_{x,I0} & V_{y,I0} & V_{z,I0} \end{bmatrix}$$

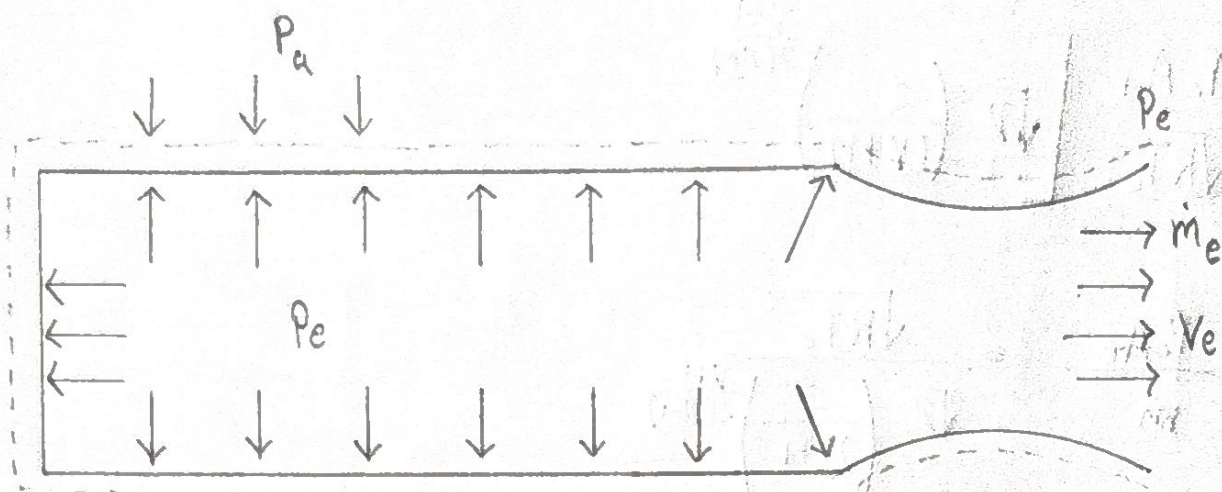
At $t = t_0$,

$$\Rightarrow V_{I,0} = \vec{\omega}_E \times \vec{r}_{I,0}$$

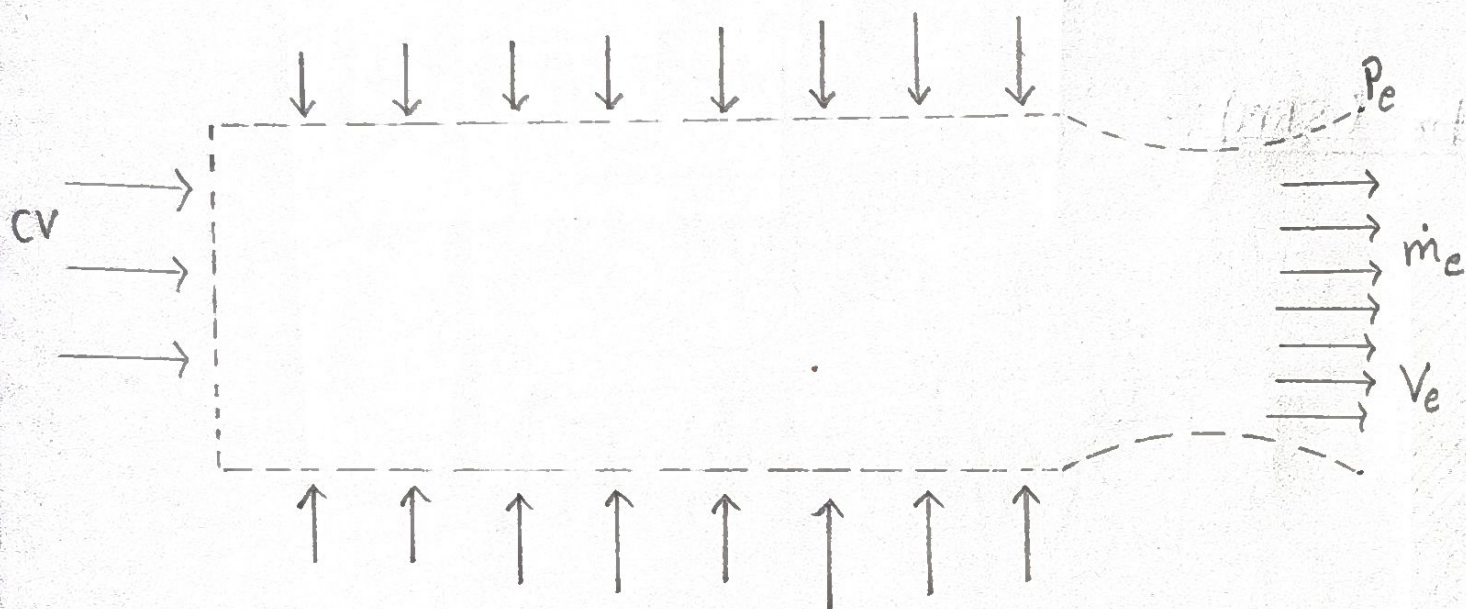


Chapter - 3 : Rocket Propulsion

Date :- 20/02/2026



$\left\{ P_a = \text{Ambient Pressure} \right\}$



$$\Rightarrow F = \dot{m}_e V_e + (P_e - P_a) A_e$$

— Adiabatic Energy Equation

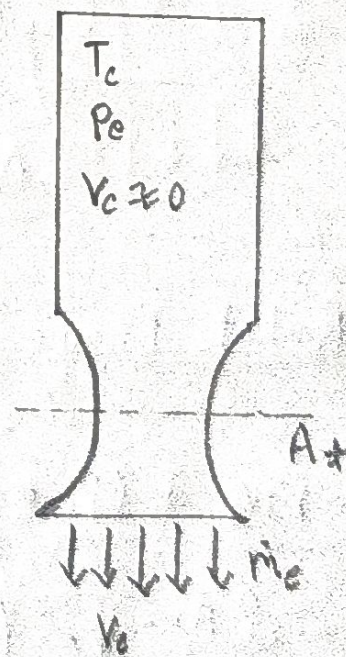
$$\Rightarrow \underbrace{\frac{V_o^2}{2}}_{V_o \approx 0} + C_p T_c = \frac{V_e^2}{2} + C_p T_e$$

$$\Rightarrow V_e = \sqrt{2 C_p (T_c - T_e)}$$

$\frac{\gamma R}{\gamma - 1} \rightarrow \frac{R_u}{M_w}$ — molecular weight

$$\Rightarrow F = \dot{m}_e V_e \rightarrow \underbrace{C^*}_{\text{characteristic velocity}} C_f$$

characteristic velocity only depend on propellant condition.

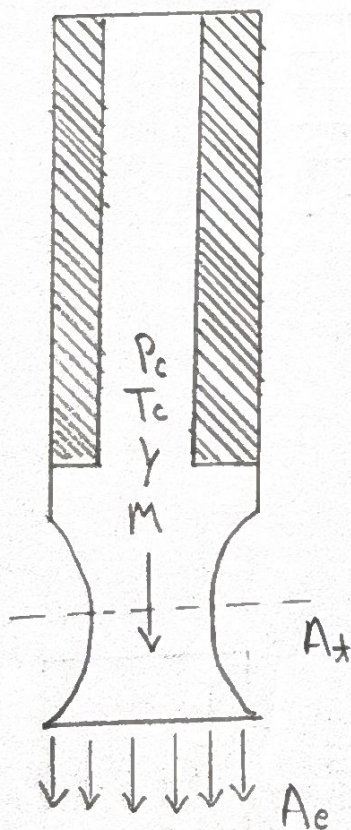


Choked Mass flow rate equation

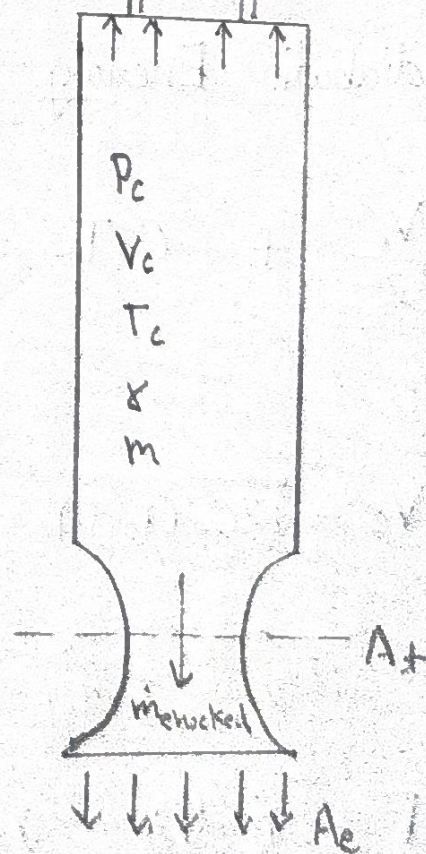
$$\Rightarrow \dot{m} = \frac{P_c A_t}{\sqrt{RT_c}} \left[\sqrt{\gamma} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \right]$$

$$\Rightarrow C^* = \frac{P_c A_t}{\dot{m}} = \frac{\sqrt{RT_c}}{\sqrt{\gamma} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}}}$$

Solid Rocket Motor (SRM) :-



Oxydizer → fuel ←



Liquid Rocket Engine (LRE) :-

$$\Rightarrow f = \dot{m} v_e + (P_e - P_a) A_e$$

$$\Rightarrow \dot{m} = \frac{P_c A_t}{\sqrt{RT_c}} \left[\sqrt{\gamma} \left(\frac{2}{\gamma+1} \right)^{\frac{\gamma+1}{2(\gamma-1)}} \right]$$

$$\Rightarrow \dot{m} = \frac{P_c A_t}{\sqrt{RT_c}} C^*$$

$$\Rightarrow C^* = \frac{P_c A^*}{\dot{m}} = \frac{\sqrt{RT_c}}{\Gamma}$$

Energy Equation:-

$$\Rightarrow \underbrace{\frac{V_{\text{chamber}}^2}{2}}_{\text{negligible}} + C_p T_c = \frac{V_e^2}{2} + C_p T_e$$

$$\Rightarrow \therefore V_e = \sqrt{2 C_p (T_c - T_e)}$$

$$\Rightarrow V_e = \sqrt{\frac{2 \gamma R}{\gamma - 1} T_c \left[1 - \left(\frac{P_e}{P_c} \right)^{\frac{\gamma - 1}{\gamma}} \right]}$$

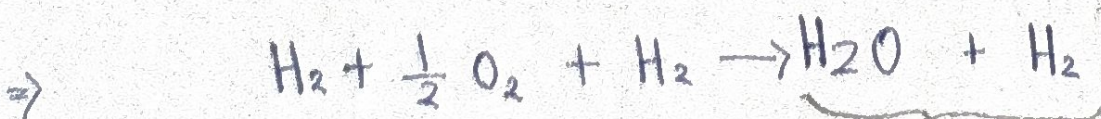
$$\Rightarrow V_e = \sqrt{\frac{2 \gamma}{\gamma - 1} \left(\frac{R_u}{m} \right) T_c \left[1 - \left(\frac{P_e}{P_c} \right)^{\frac{\gamma - 1}{\gamma}} \right]}$$

$$\Rightarrow V_e = \sqrt{RT_c} \cdot \sqrt{\frac{2 \gamma}{\gamma - 1} \left(1 - \left(\frac{P_e}{P_o} \right)^{\frac{\gamma - 1}{\gamma}} \right)}$$

from $e^* = \sqrt{RT_c} = C^* \Gamma$

$$\Rightarrow V_e = C^* \Gamma \sqrt{\frac{2 \gamma}{\gamma - 1} \left(1 - \left(\frac{P_e}{P_c} \right)^{\frac{\gamma - 1}{\gamma}} \right)}$$

$$\Rightarrow \boxed{V_e = C^* \cdot C_f}$$



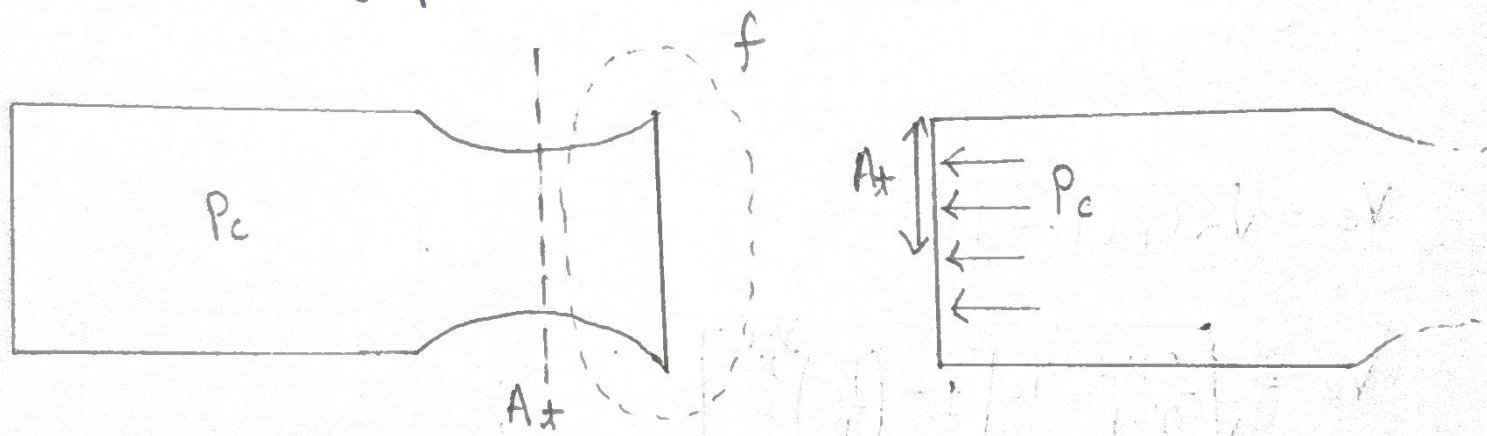
effectively molecular weight will decrease $\left(\frac{R_u}{m} \right) \uparrow$

⇒ In Ideal Case Thrust,

⇒ $P_e = P_a$

⇒ $F = m_e V_e = \dot{m} C^* \bar{C}_F$

⇒ $C_F = \frac{F}{P_c A_t} \left\{ \text{This will be higher than 1} \right\}$



⇒ $\frac{A_e}{A_t} \rightarrow \text{function } \frac{P_e}{P_c}, \bar{C}_F$

⇒ $P_e V_e A_e = P_t V_t A_t$

⇒ $\frac{A_e}{A_t} = \frac{\Gamma^2}{\left(\frac{P_e}{P_c}\right)^{\frac{1}{\gamma}} \cdot C_F}$

Thrust Coefficient

⇒ $C_t^* = P_e = P_a$

⇒ $C_t^* = \Gamma \sqrt{\frac{2\gamma}{\gamma-1} \left(1 - \left(\frac{P_e}{P_c}\right)^{\frac{\gamma-1}{\gamma}} \right)}$

Propulsion System Design and Sizing :-

$$\Rightarrow M_0 = 1000 \text{ kg (Without boosters)}$$

(This will be our Second Assignment)

$$\Rightarrow LV \rightarrow \frac{T_0}{W} = 1.2$$

$$\Rightarrow T_0 = 1.2 \times 1000 \times 9.8 \\ = 11760 \text{ N}$$

$$\Rightarrow F = 11760 \text{ N}$$

$$\Rightarrow SRM \Rightarrow I_{sp} \sim 230 \text{ s} \rightarrow 230 \times 9.8 \\ \approx 2300 \text{ m/s}$$

$$\Rightarrow \dot{m}_e = \frac{11760}{2300} = 2.11 \text{ kg}$$

$$\Rightarrow \frac{P_c}{P_e} = '30 \text{ to } 50'$$

for

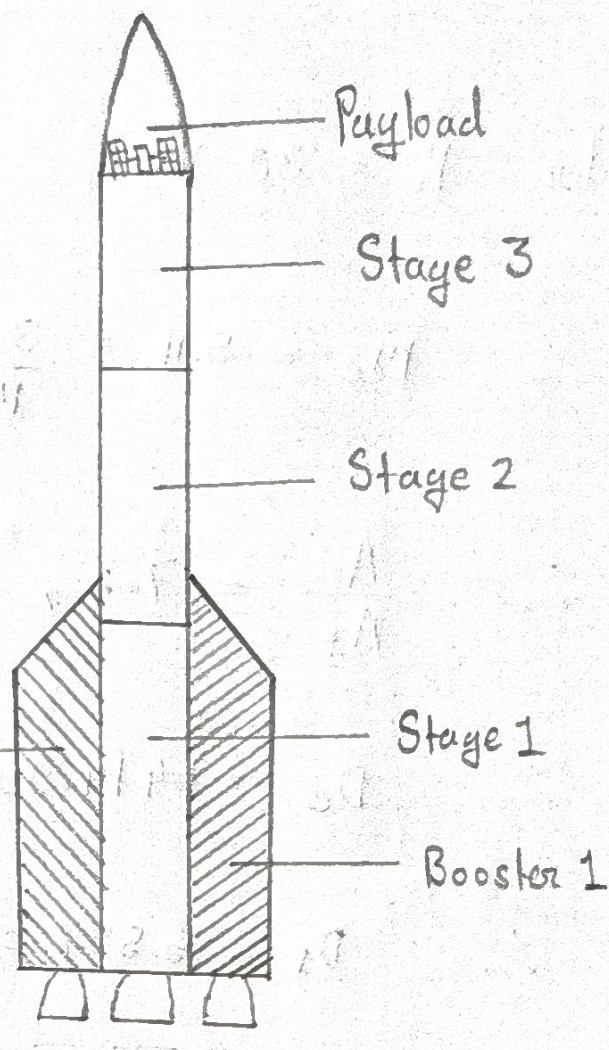
$$\Rightarrow \frac{P_c}{P_e} = 50$$

$$\Rightarrow \gamma = 1.2$$

$$\Rightarrow C_{f'} = \Gamma \sqrt{\frac{2\gamma}{\gamma-1} \left(1 - \left(\frac{P_e}{P_c} \right)^{\frac{\gamma-1}{\gamma}} \right)}$$

$$\Rightarrow C_f = \Gamma \sqrt{\frac{2.4}{0.2} \left(1 - \left(\frac{1}{50} \right)^{\frac{0.2}{1.2}} \right)}$$

$$\Rightarrow C_f = 1.61$$



$$\Gamma = \sqrt{1.2} \left(\frac{2}{2.2} \right)^{\frac{2.2}{2(0.2)}} = 0.64826$$

$C_{f'}$ will approach 2 as $\frac{P_c}{P_e} \rightarrow \infty$

$$A_t = 0.001468 \text{ m}^2 \rightarrow D_t \rightarrow 43 \text{ cm}$$

$$A_e = 7.2 \times A_t = 0.0705 \text{ m}^2 \rightarrow D_e \rightarrow 115.7 \text{ cm}$$

$$\dot{m}_e = \frac{P_c A_t}{C^*}$$

$$\therefore \frac{P_c \times A_t}{430.35} = 5.11$$

Propulsion System Design and Sizing :-

Date :- 25/02/2026

Continued -

Consider $P_c = 30$ bar

$$\Rightarrow \dot{m}_e = 5.11 = \frac{30 \times 10^5 \cdot A_t}{1430.35} \Rightarrow A_t = 0.002436 \text{ m}^2$$

$$\Rightarrow \therefore \frac{A_e}{A_t} = 7.2, A_e = 0.01754 \text{ m}^2$$

$$\Rightarrow \therefore D_e = 14.99 \text{ cm} \quad (11.57 \text{ cm for } P_c = 50 \text{ bar})$$

$$\Rightarrow \therefore D_t = 5.557 \text{ cm} \quad (4.3 \text{ cm for } P_c = 50 \text{ bar})$$

$$\Rightarrow \frac{P_c}{P_e} = 50 \Rightarrow P_e = 0.6 \text{ bar}$$

Summerfield Criteria $\rightarrow$ flow becomes overexpanded if $\frac{P_e}{P_a} < 0.4$
{flow separation}

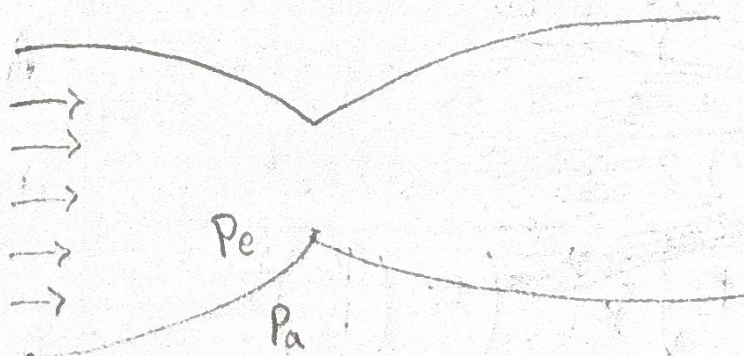
$$\Rightarrow C_f = C_f^* + \left[\left(\frac{P_e}{P_c} \right) - \left(\frac{P_a}{P_c} \right) \right] \left(\frac{A_e}{A_t} \right)$$

$$\Rightarrow = 1.61 + \left(\frac{1}{50} - \frac{1}{30} \right) 7.2$$

$$\Rightarrow = 1.61 + (-0.0133) \times 7.2$$

$$\Rightarrow = 1.514$$

$$\Rightarrow \frac{P_e}{P_a} < 0.4$$



Over-expanded nozzle

$$\Rightarrow \frac{f}{P_c A_t} = \frac{\dot{m} e V_e}{A_c A_t} + \frac{(P_e - P_a)}{P_c A_t} A_c$$

$$\Rightarrow = C_f^* + \left(\frac{P_e}{P_c} - \frac{P_a}{P_c} \right) \cdot \frac{A_c}{A_t}$$

C_f^* will not change because P_c is changing but P_e/P_c is not changing

$$\Rightarrow F = \dot{m}_e \times C^* \times C_f$$

$$\Rightarrow = 5.11 \times 1430.35 \times 1.514$$

$$\Rightarrow = 11298 \text{ N}$$

$$\Rightarrow \text{for } \frac{T}{W} = 1.2$$

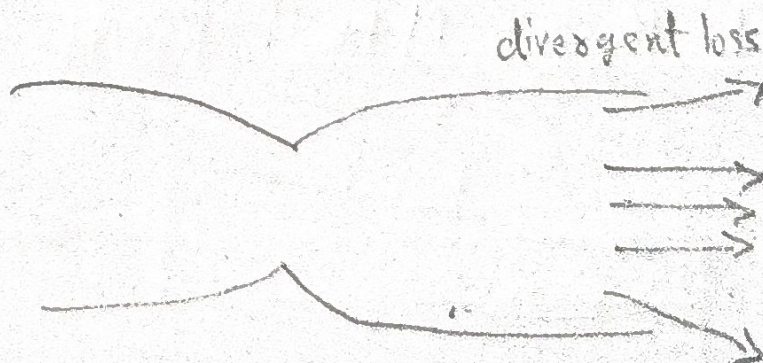
$$\Rightarrow f = 117.60$$

Consider $P_c = 30 \text{ bar}$

$$\Rightarrow 11760 = \dot{m}_{\text{now}} \times C^* \times C_f$$

$$\Rightarrow \dot{m} = 5.47 = \frac{P_c A_t}{C^*} \Rightarrow A_t = 0.0026 \text{ m}^2 \rightarrow D_t = 5.8 \text{ cm}$$

$$\Rightarrow C_{\text{efficiency}}^* = \frac{C_{\text{actual}}^*}{C_{\text{theoretical}}^*}$$



$$\Rightarrow C_{f \text{ efficiency}} = \frac{C_{f \text{ actual}}}{C_{f \text{ theoretical}}}$$

for $P_c = 30 \text{ bar}$

$\Rightarrow C_{n_{\text{eff}}}^* = 0.9, \quad C_{f_{n_{\text{eff}}}} = 0.9$

$\Rightarrow C_{f_{\text{act}}} = 0.9 \times 1.51 = 1.359$

$\Rightarrow C_{\text{act}}^* = 0.9 \times 1430 = 1287$

$\Rightarrow \therefore 11760 = \dot{m}_{\text{new}} \times C_{\text{act}}^* \times C_{f_{\text{act}}}$

$\Rightarrow \therefore \boxed{\dot{m}_{\text{new}} = 6.72}$

$\dot{m} = \frac{P_c A_t}{C_{\text{act}}^*}$

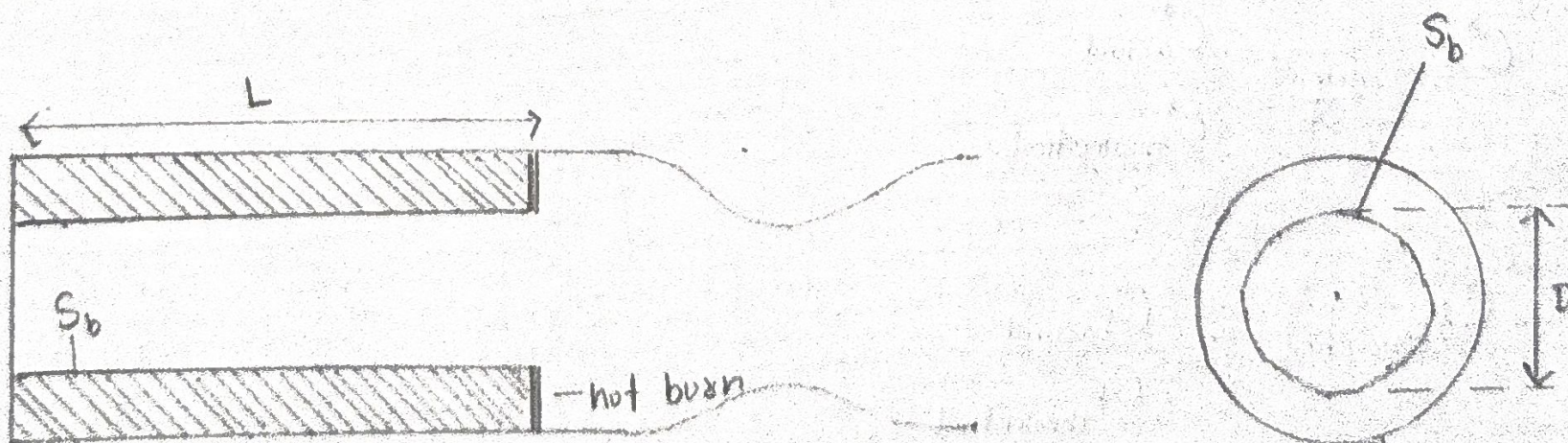
$\Rightarrow A_{t_{\text{new}}} = 0.00288 \text{ m}^2$

$\Rightarrow A_e = 7.2 \times A_{t_{\text{new}}} = 0.020736$

$\Rightarrow D_{t_{\text{new}}} = 6.05 \text{ cm}$

$\Rightarrow D_{e_{\text{new}}} = 16.25$

Solid Rocket Motor :- {SRM}



$$S_b = \pi DL$$

$$\dot{m}_{\text{generated}} = \dot{m}_{\text{nozzle}}$$

$$\underset{p^n}{\rho_b g_1 S_b} = \frac{\rho_c A_t}{c^*}$$

$$S_b a p^n g_p = \frac{\rho_c A_t}{c^*}$$

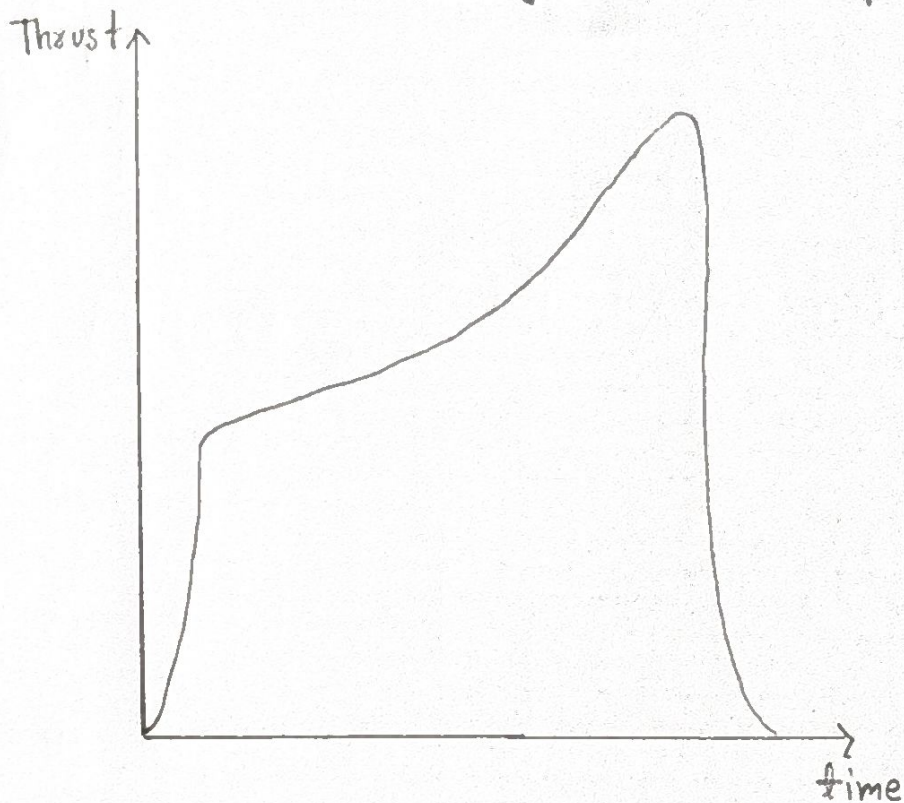
$$P_c = \left\{ g_p a c^* \left(\frac{S_b}{A_t} \right) \right\}^{\frac{1}{1-n}}$$

$a \& n$, we will get by the experiments for a Selected propellant Combination

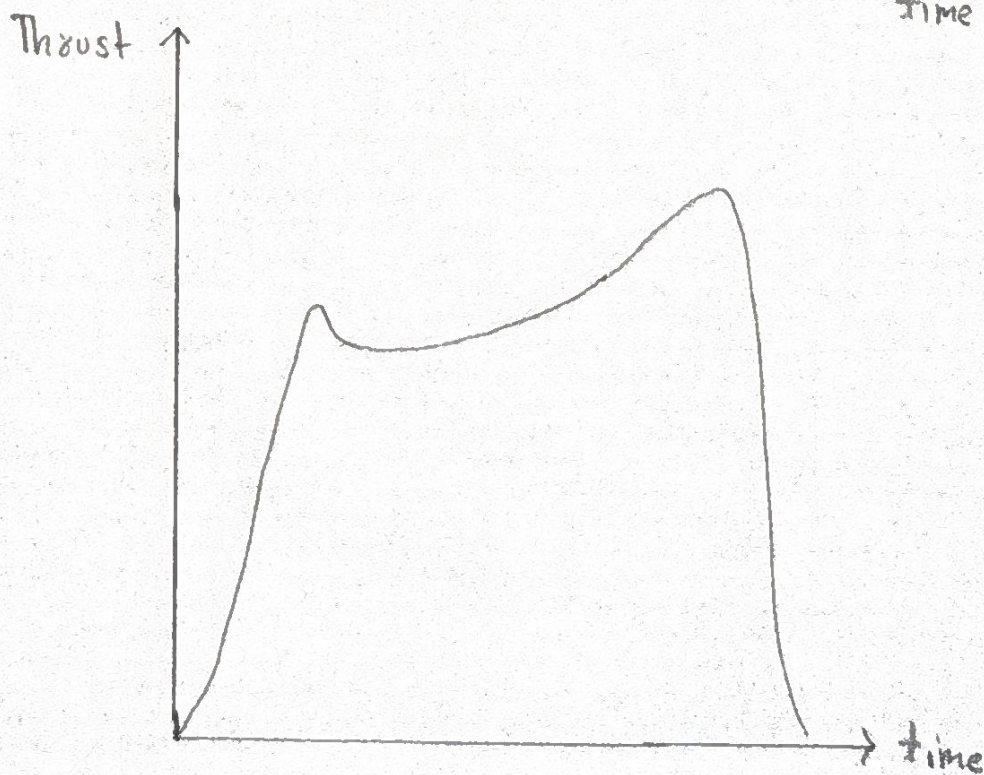
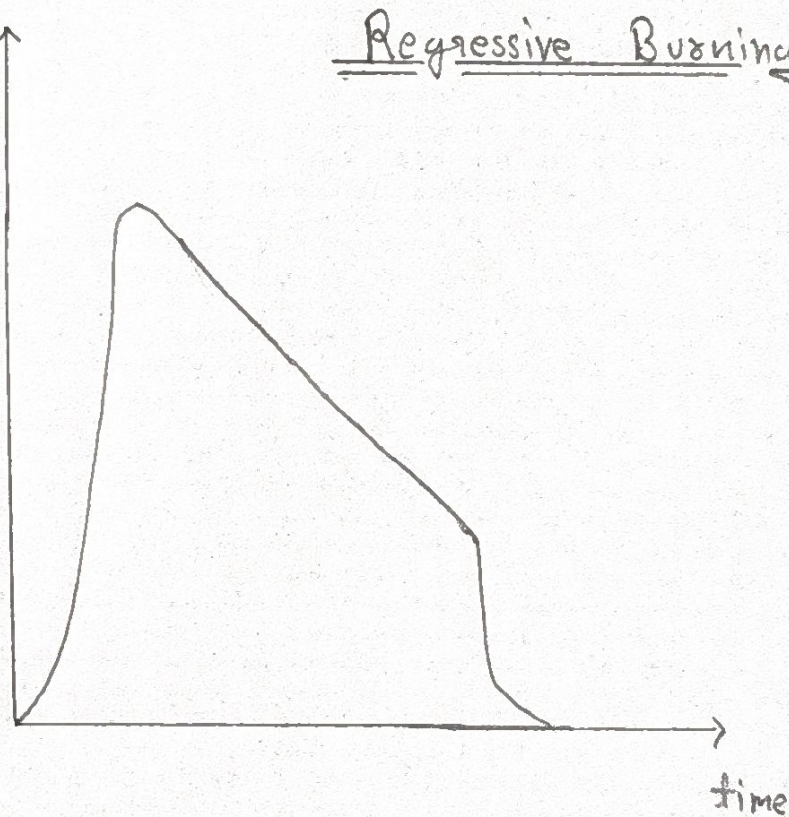
$n > 1$ and $n \approx 1$ will Cause the motor to become Unstable

S_o , $n \rightarrow 0.2 - 0.5$ is desirable.

Progressive Burning



Regressive Burning



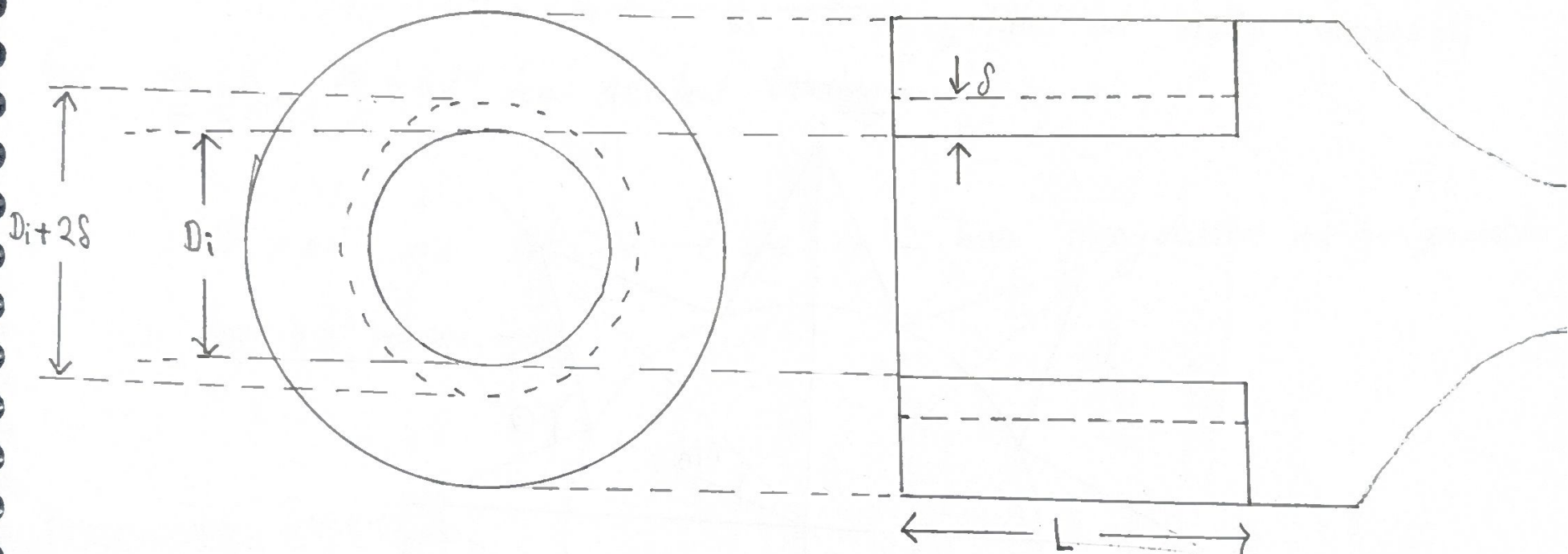
Date: 26/02/2026

⇒

$$P_c = \left\{ \rho_p a c^* \frac{P_b}{A_t} \right\}$$

⇒

$$P_c = C_f P_c A_t$$



P_c vs time provides thrust

⇒ A_t

$$D = D_i \rightarrow P_c = \left\{ \rho_p c_d c^* \frac{\pi D_i L}{A_t} \right\}^{\frac{1}{1-n}}$$

⇒ A_t

$$D = D_i + 2\delta \rightarrow P_{c_{D_i+2\delta}} = \left\{ \rho_p c_d c^* \frac{\pi (D_i + 2\delta) L}{A_t} \right\}^{\frac{1}{1-n}}$$

$\delta \rightarrow \Delta t$

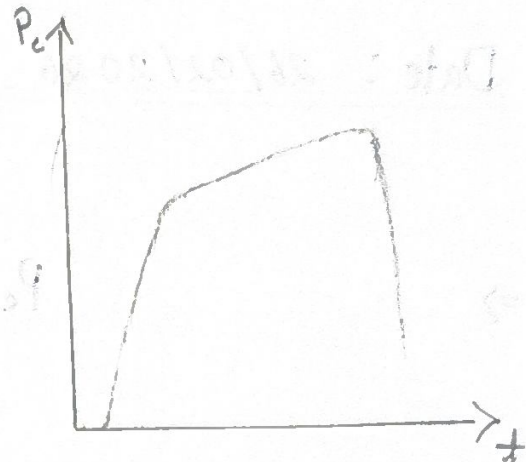
⇒

$$\eta = \frac{\delta}{\Delta t}$$

We are selecting the δ (in mm). δ is regression

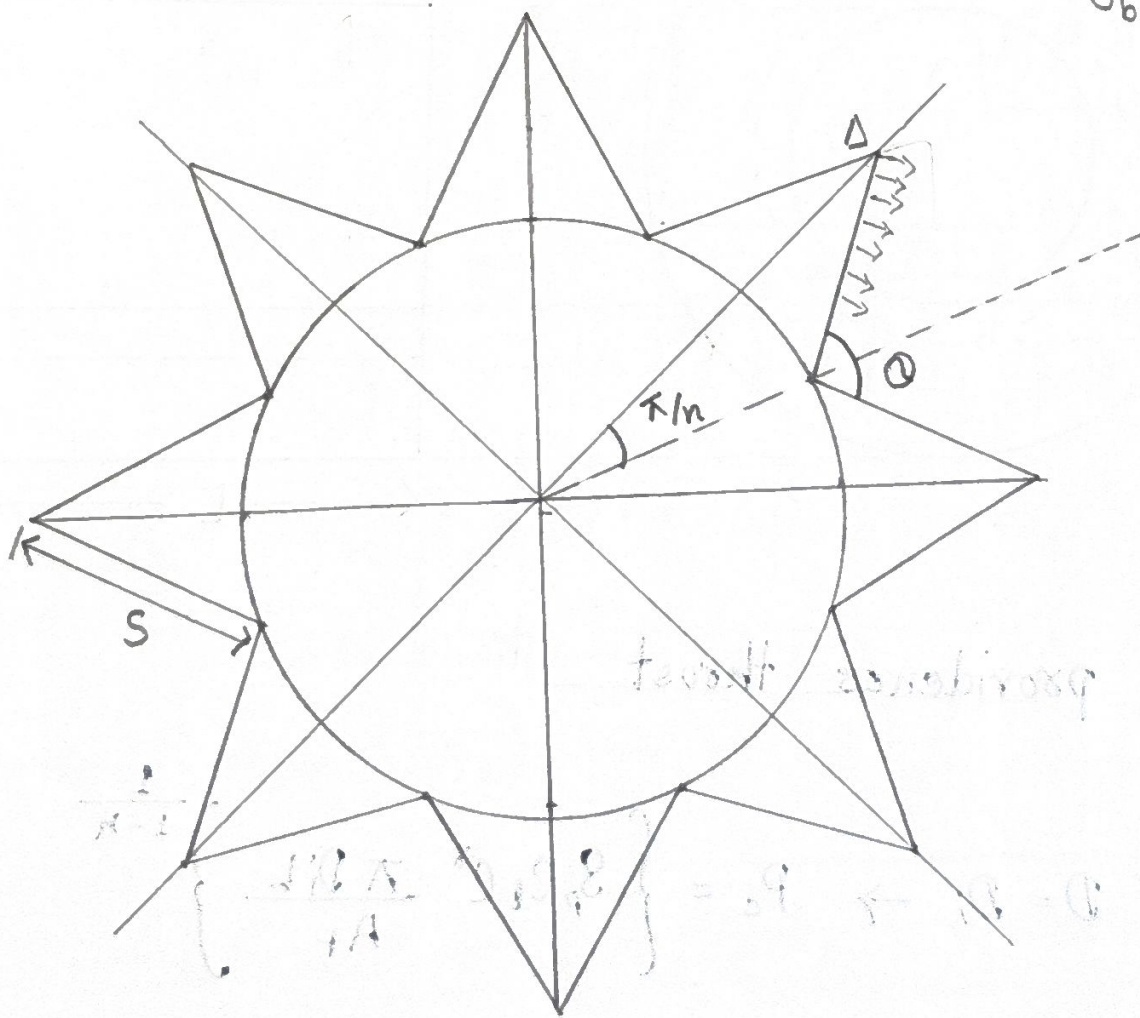
$$\Rightarrow \therefore \Delta t = \frac{\delta}{\eta} = \frac{\delta}{a \left(\frac{P_{cD1} + P_{c(D1+2\delta)}}{2} \right)}$$

avg. pressure



The P_c vs time graph does not start exactly at $t = 0$ because after choking and ignition, a small amount of time passes before pressure starts to building up.

$$S_b = 2nS.L$$



Radial Burning $\rightarrow$ burning rate will be $\perp$ to the Surface

$$t_0 \rightarrow S$$

$$t_0 + \Delta t \rightarrow$$

$$\Delta \left[\frac{\pi}{2} + \frac{\pi}{n} - \frac{\theta}{2} \right] + S - \Delta \cot \frac{\theta}{2}$$

Neutral Burning :- $S_b = \text{Constant}$ (Constant thrust throughout)

$$\Rightarrow S = \Delta \left(\frac{\pi}{2} + \frac{\pi}{n} - \frac{\phi}{2} \right) - \Delta \cot \frac{\phi}{2} + S$$

$$\Rightarrow \boxed{\cot \frac{\phi}{2} = \frac{\pi}{2} + \frac{\pi}{n} - \frac{\phi}{2}}$$

for $n=6$, $\phi=67^\circ$ for neutral burning.

$\Rightarrow \phi > 67$ or $\phi < 67 \rightarrow$ we will have progressive or regressive burning respectively.

Progressive Burning

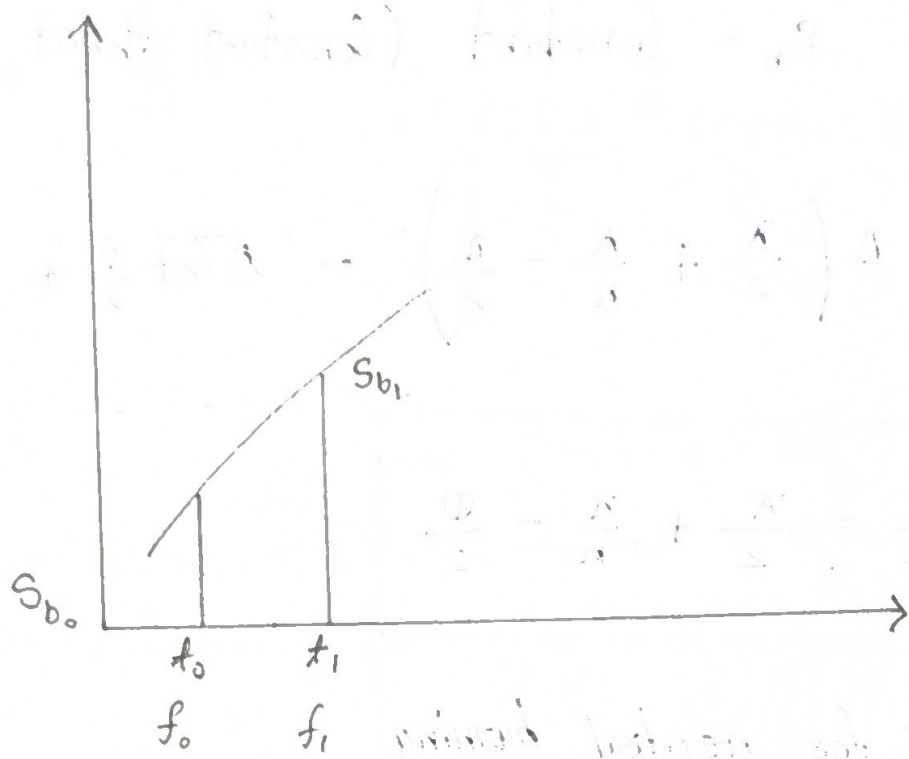
$$\Rightarrow \Delta \left[\frac{\pi}{2} + \frac{\pi}{n} - \frac{\phi}{2} \right] + S - \Delta \cot \frac{\phi}{2} > S$$

$$\Rightarrow \boxed{\cot \frac{\phi}{2} < \left[\frac{\pi}{2} + \frac{\pi}{n} - \frac{\phi}{2} \right]}$$

Regressive Burning

$$\Rightarrow \Delta \left(\frac{\pi}{2} + \frac{\pi}{n} - \frac{\phi}{2} \right) + S - \Delta \cot \frac{\phi}{2} < S$$

$$\Rightarrow \boxed{\cot \frac{\phi}{2} > \left[\frac{\pi}{2} + \frac{\pi}{n} - \frac{\phi}{2} \right]}$$



→ $\dot{m}$ is known. I_{sp} is max. given M.R. at the time of lift off.

$$\Rightarrow f = \dot{m}_e v_e$$

$$\Rightarrow m_p = \dot{m}_e \times t_b$$

$$\Rightarrow \dot{m}_p = \dot{m}_{ox} + \dot{m}_f$$

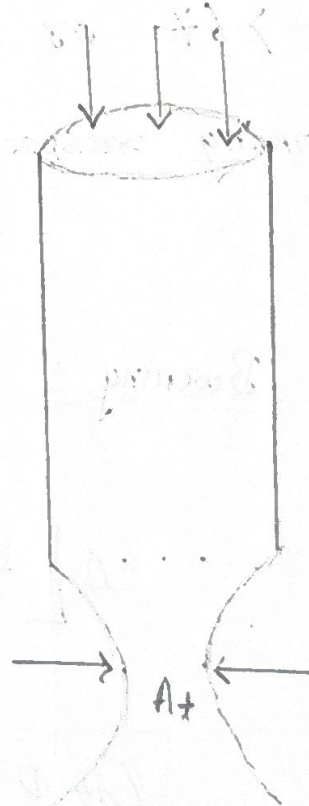
$$\Rightarrow M_R = \frac{M_{ox}}{m_f}$$

→ Characteristic Length

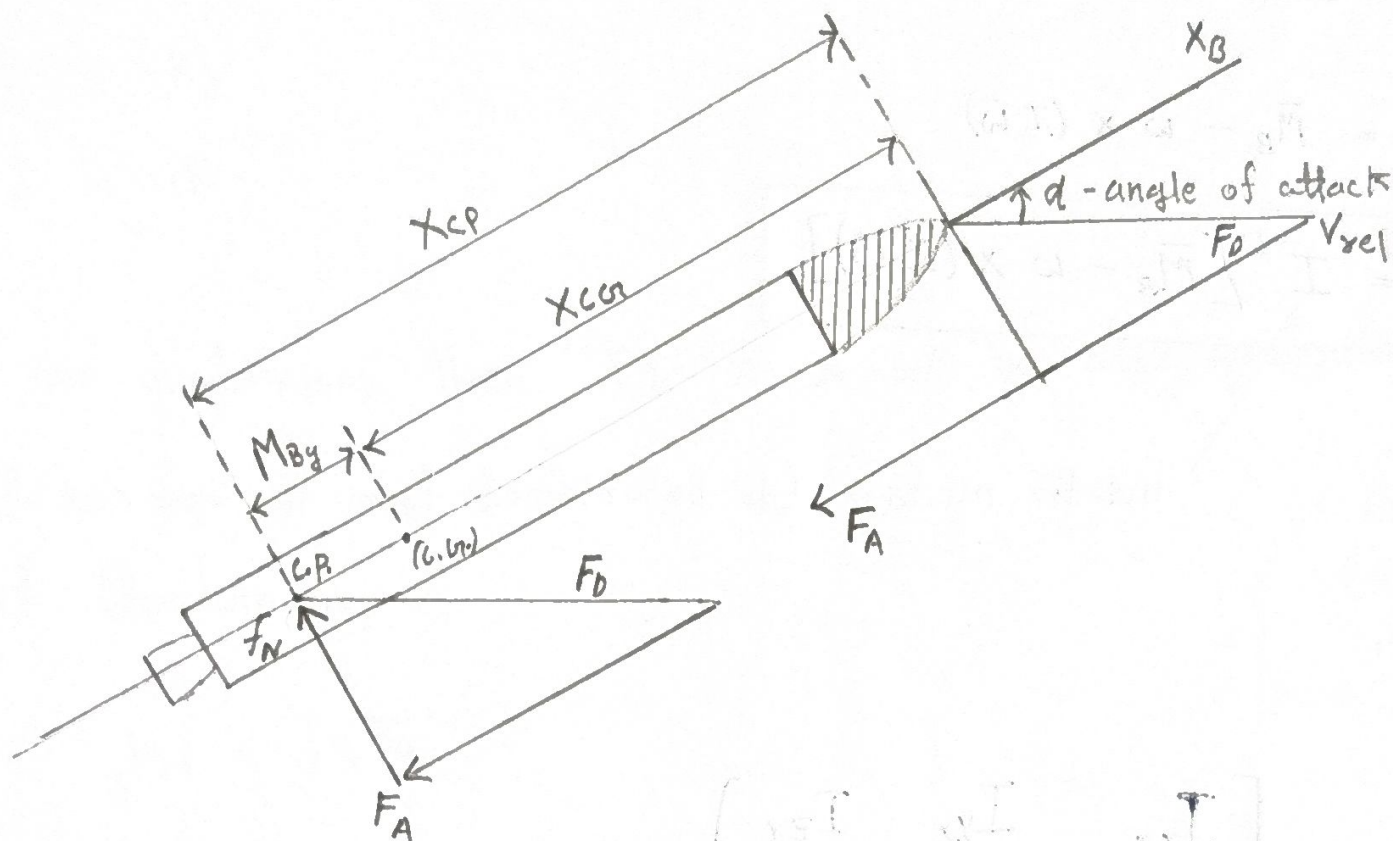
$$\Rightarrow L^* = \frac{v_e}{A_t}$$

$$\Rightarrow \rho = \text{Constant}$$

V should not be too huge but it should be large enough for mixing.



Chapter 4 :- 6 DoF Trajectory Simulation - Launch Vehicle



$$\Rightarrow C_D = \text{given}, \quad f_D = \frac{1}{2} \rho V_{rel}^2 C_D S_{ref}$$

$$\Rightarrow M_{By} = (X_{cp} - X_{cn}) f_N$$

$C_n \rightarrow$ normal force coefficient

$$\Rightarrow M_{By} = -(X_{cp} - X_{cn}) \cdot \frac{1}{2} \rho V_{rel}^2 S_{ref} C_n$$

How to bring C_p down practically? $\rightarrow$ attaching fins, Pressure distribution will be converting into force. if we make it highly stable the Controlling is difficult.

$$\Rightarrow \sum \bar{M} = \dot{\bar{H}}|_{rel} + \omega \times \bar{H}$$

$$\Rightarrow \frac{d\bar{B}}{dt} = \frac{d\bar{B}}{dt}|_{rel} + \bar{\omega} \times \bar{B}$$

$$\Rightarrow \bar{M}_B = I\dot{\bar{\omega}} + \bar{\omega} \times (I\bar{\omega})$$

$$\Rightarrow I\dot{\bar{\omega}} = \bar{M}_B - \bar{\omega} \times (I\bar{\omega})$$

$$\Rightarrow \boxed{\dot{\bar{\omega}} = I^{-1} [\bar{M}_B - \bar{\omega} \times (I\bar{\omega})]}$$

integrate to Set $\bar{\omega}$
using Forward
Euler / RK4

inertia matrix (to be obtained from the geometry of the vehicle)

$$\Rightarrow I = \begin{bmatrix} I_{xx} & I_{yx} & I_{zx} \\ I_{xy} & I_{yy} & I_{zy} \\ -I_{zx} & I_{yz} & I_{zz} \end{bmatrix}$$

$$\Rightarrow I = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

p - Pitch rate

q - Yaw rate

r - Roll rate

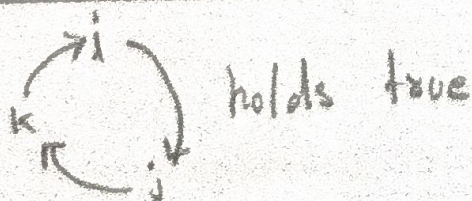
Kind of related to $\dot{\psi}, \dot{\phi}, \dot{\theta}$

★ Quaternions

$$\Rightarrow q = q_0 + \bar{q} = q_0 + \underbrace{i q_1 + j q_2 + k q_3}_{\bar{q}} \quad \begin{matrix} i = (1, 0, 0) \\ j = (0, 1, 0) \\ k = (0, 0, 1) \end{matrix}$$

here,

$$\Rightarrow \boxed{i^2 = j^2 = k^2 = -1}$$



$$\Rightarrow ij = k = -ji$$

$$\Rightarrow jk = i = -kj$$

$$\Rightarrow ki = j = -ik$$

if p, q are quaternions then $pq = r$, r will be also quaternion

Norm of Quaternion,

$$\Rightarrow |q| = \sqrt{q^* \cdot q}$$

$$\Rightarrow q^* = q_0 - \bar{q} = \text{Complex Conjugate}$$

if $\sqrt{q^* \cdot q} = 1$, q is called unit quaternion

$$\Rightarrow q^* q = 1$$

$$\Rightarrow \therefore (q_0 - \bar{q})(q_0 + \bar{q}) = 1$$

$$\Rightarrow \therefore q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

$$\Rightarrow q_0^2 + |\bar{q}|^2 = 1$$

$$\&, \cos^2 \theta + \sin^2 \theta = 1 \quad \text{So we can write,}$$

$$\Rightarrow q_0 = \cos \theta, \quad q = \sin \theta : |\bar{q}|^2 = \sin^2 \theta, |\bar{q}| = \sin \theta$$

And therefore, if we take $|\bar{q}| = \cos \theta$ and $q = \sin \theta$, then by arranging these equation to obtain q , we get the following result.

$$\Rightarrow q = q_0 + \bar{q}$$

$$\Rightarrow = \cos \theta + \hat{u} \sin \theta$$

Unit Vector along $\bar{q}$

$$\Rightarrow \hat{u} = \frac{q}{|q|}$$

$$\Rightarrow \hat{u} = \frac{\bar{q}}{\sin \theta} \quad \text{hence } \bar{q} = \hat{u} \sin \theta$$

put $(-\theta)$ then

$$\Rightarrow q = \cos(-\theta) + \hat{u} \sin(-\theta)$$

$$\Rightarrow = \cos \theta - \hat{u} \sin \theta$$

$$\Rightarrow q_0 - \bar{q} = q^*$$

Date: 19/03/2026

Repeat of previous class

★ Quaternions:-

$$q = q_0 + \bar{q}$$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

$$\text{norm} = \sqrt{q^* \cdot q}$$

$$q^* = q_0 - \bar{q} \text{ (Complex Conjugate)}$$

$$\text{norm} = \sqrt{q^* \cdot q} = \sqrt{(q_0 - \bar{q})(q_0 + \bar{q})} = \sqrt{q_0^2 + |\bar{q}|^2}$$

★ Unit quaternion

$$\text{norm} = 1$$
$$\sqrt{q^* \cdot q} = \sqrt{q_0^2 + |\bar{q}|^2} = 1$$

$$q_0^2 + |\bar{q}|^2 = 1$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$q_0 = \cos \theta$$

$$|\bar{q}| = \sin \theta$$

$$\rightarrow q = \hat{u} |\bar{q}|, \quad \hat{u} = \frac{\bar{q}}{|\bar{q}|} = \frac{\bar{q}}{\sin \theta}$$

for a unit quaternion,

$$q = q_0 + \bar{q} = q_0 + \hat{u} |\bar{q}| = \cos \theta + \hat{u} \sin \theta$$

$$\rightarrow q \cos \theta + \hat{u} \sin \theta$$

$$q^* = \cos(-\theta) + \hat{u} \sin(-\theta) = \cos\theta - \hat{u} \sin\theta$$

lets say,

$$\hat{u} = 0i + 0j + 1k$$

Consider θ to be very small

$$q = \cos\theta + \hat{u} \sin\theta$$

$$= 1 + k\theta$$

$$q^* = 1 - k\theta$$

Consider a Vector,

$$V = i + 0j + 0k$$

→ lets us perform the operation $qVq^* = W$

$$W = (1 + k\theta)(i)(1 - k\theta)$$

$$= (i + k\theta)(1 - k\theta)$$

$$= i - i k\theta + k\theta - k\theta^2$$

$$= i + 2\theta j - i\theta^2$$

$$\text{let } W = i + 2\theta j$$

mathematically it means,

i Vector rotated by an angle 2θ with respect to

axis k.

Theorem:- For any unit vector quaternion $q = q_0 + \bar{q} = \cos \theta + \hat{n} \sin \theta$ and for any vector $V \in \mathbb{R}^3$, action $L_q(\bar{V}) = q \bar{V} q^*$, may be geometrically interpreted as a rotation of the vector V through an angle 2θ about $\bar{q}$ as the axis of rotation.

$$W = q V q^* = (2q_0^2 - 1)\bar{V} + 2(\bar{q} \cdot \bar{V})\bar{q} + 2q_0(\bar{q} \times \bar{V})$$

$$\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = W = \begin{bmatrix} (2q_0^2 - 1 + 2q_1^2) & (2q_1q_2 - 2q_0q_3) & (2q_1q_3 + 2q_0q_2) \\ (2q_1q_2 + 2q_0q_3) & (2q_0^2 - 1 + 2q_2^2) & (2q_2q_3 - 2q_0q_1) \\ (2q_2q_3 - 2q_0q_1) & (2q_1q_3 + 2q_0q_2) & (2q_0^2 + 2q_3^2) \end{bmatrix}$$

① matrix

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$L_q(\bar{V}) = q \bar{V} q^*$$

What about $q^* \bar{V} q$? $\Rightarrow$ rotation will be of -2θ instead of 2θ
let $\pi = qP$, q is rotated, P is rotated hence π is also be rotated. Combinedly,

★ Euler Angles to quaternions:-

$$q_y = \cos \frac{\phi}{2} + j \sin \frac{\phi}{2}$$

$$q_z = \cos \frac{\psi}{2} + k \sin \frac{\psi}{2}$$

$$q_x = \cos \frac{\phi}{2} + i \sin \frac{\phi}{2}$$

as it gives doubles the angle
we select it as $\frac{\phi}{2}, \frac{\psi}{2}, \frac{\phi}{2}$.

$$q = q_y q_z q_x$$

to get $q_0, q_1, q_2, q_3 \rightarrow$ multiply and compose.

$$q = \left(\cos \frac{\Theta}{2} + j \sin \frac{\Theta}{2} \right) \left(\cos \frac{\Psi}{2} + k \sin \frac{\Psi}{2} \right) \left(\cos \frac{\Phi}{2} + i \sin \frac{\Phi}{2} \right)$$

$$q_0 = \cos \frac{\Theta}{2} \cos \frac{\Psi}{2} \cos \frac{\Phi}{2} - \sin \frac{\Theta}{2} \sin \frac{\Psi}{2} \sin \frac{\Phi}{2}$$

$$q_1 = \cos \frac{\Theta}{2} \cos \frac{\Psi}{2} \sin \frac{\Phi}{2} + \sin \frac{\Theta}{2} \sin \frac{\Psi}{2} \cos \frac{\Phi}{2}$$

$$q_2 = \cos \frac{\Theta}{2} \sin \frac{\Psi}{2} \sin \frac{\Phi}{2} + \cos \frac{\Psi}{2} \cos \frac{\Phi}{2} \sin \frac{\Theta}{2}$$

$$q_3 = \cos \frac{\Theta}{2} \cos \frac{\Phi}{2} \sin \frac{\Psi}{2} - \sin \frac{\Theta}{2} \cos \frac{\Psi}{2} \sin \frac{\Phi}{2}$$

$${}^B R_{LPI} = R_1(\phi) R_3(\psi) R_2(\theta)$$

$$\begin{bmatrix} X_B \\ Y_B \\ Z_B \end{bmatrix} = \begin{bmatrix} \cos \psi \cos \theta & \sin \psi & -\cos \psi \sin \theta \\ - & \cos \phi \cos \psi & - \\ - & -\sin \phi \cos \psi & - \end{bmatrix} \begin{bmatrix} X_L \\ Y_L \\ Z_L \end{bmatrix}$$

L B

Same rotation can be applied using $q^* e_k q, e_k, k=1,2,3$

Base Vector

here Q will become Q^T matrix.

$$\phi_{\text{roll}} = \tan^{-1} \left(\frac{-2q_2q_3 + 2q_0q_1}{q_0^2 - q_3^2 + q_2^2 - q_1^2} \right)$$

$$\psi_{\text{yaw}} = \sin^{-1} (2q_1q_2 + 2q_0q_3)$$

$$\theta_{\text{pitch}} = \tan^{-1} \left(\frac{-2q_1q_3 + 2q_0q_2}{q_1^2 - q_3^2 + q_0^2 - q_2^2} \right)$$

At time t_0 , $\theta = 90^\circ$, $\psi = 0$, $\phi = 0$

$$q_{\text{initial}} = [q_0 \ q_1 \ q_2 \ q_3]_{\text{initial}}$$

$$\dot{\bar{\omega}} = I^{-1} \{ \bar{M}_B - \bar{\omega} \times I \bar{\omega} \}$$

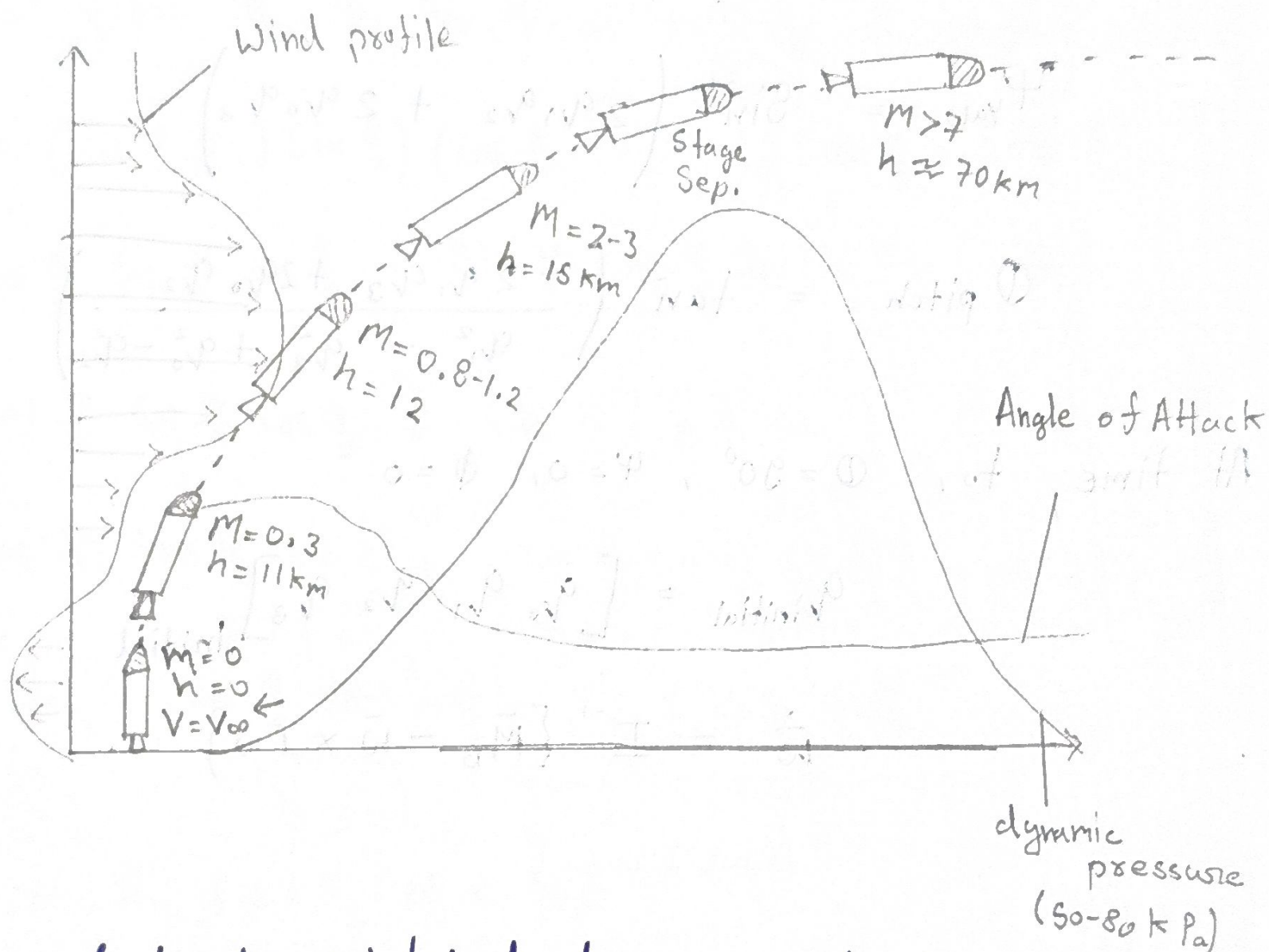
After 1 second

$$\omega_{(t_0 + 1s)} = \omega_{t_0} + \ddot{\omega} \Delta t$$

$$\frac{dq}{dt} = \frac{1}{2} \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$q_{t+\Delta t} = q_t + \frac{dq}{dt} \Delta t$$

Chapter - 5 :- Aerodynamics of LV



LV $\rightarrow$ proposed to be maintained at zero angle of Attack.

Actual flight $\rightarrow$ Wind Velocity Variation + large wind Shear $\Rightarrow$ flow incidence $1-2^\circ$.

$h \approx 40 \text{ km} \Rightarrow$ LV $\rightarrow$ hypersonic regime due to high altitude
 $q \Rightarrow$ reduced to small value.

Stage Separation, $\rightarrow q \sim 100-200 \text{ Pa}$ (not kPa)

★ Steady / Unsteady loads on LV structure,

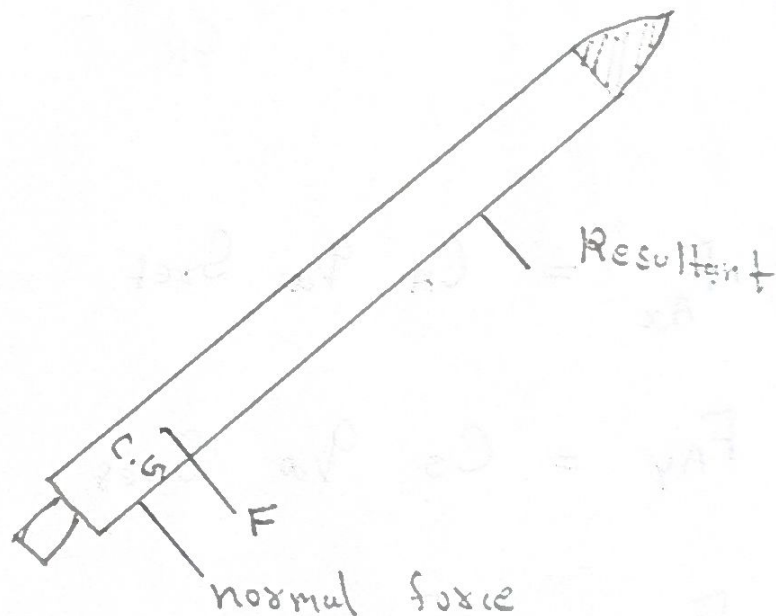
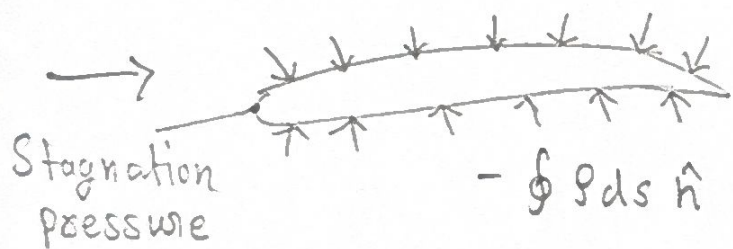
- ① Skin friction
- ② Pressure Variation
- ③ Shocks waves (wave drag)
- ④ Vortex induced
- ⑤ Interference effect.

If Shock remain attached flow Sep. will not occur and vice versa.

Unsteady flow can cause aeroelastic dynamic movement / vibrations.

Structural damage can occur.

for comparison,



$$F_A = F_s + F_p$$

force due to pressure

force due to shear.

$$\bar{M}_A = M_s + M_p$$

about
C.G.

$$F_A = f_1(\alpha, V_\infty, \rho_\infty, M_\infty, D_{ref}, C_{ref})$$

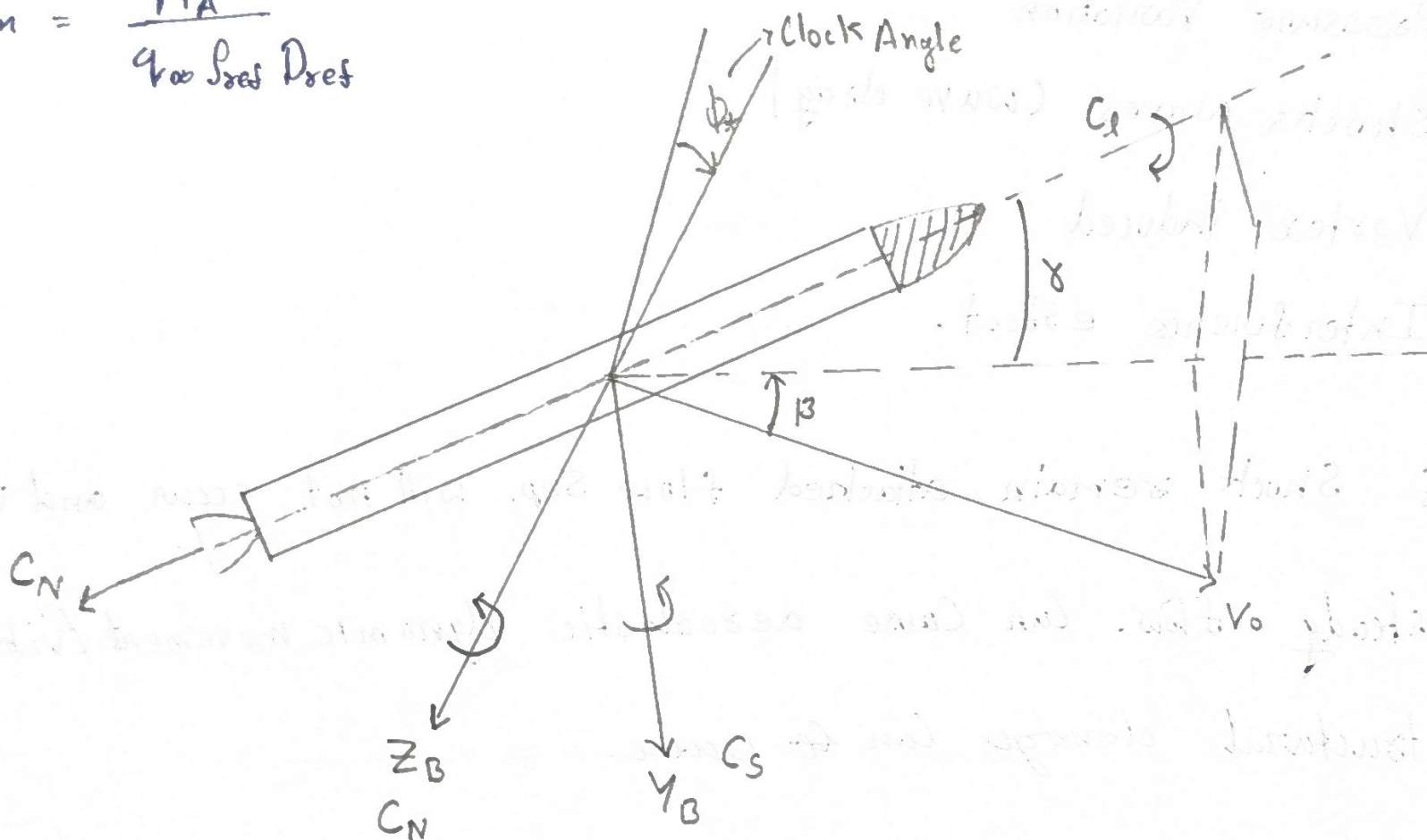
$$M_A = f_2(\alpha, V_\infty, \rho_\infty, M_\infty, D_{ref}, q_\infty)$$

$$C_f = \frac{f_A}{q_\infty S_{ref}}$$

$$C_f = f_3(\alpha, Re, M)$$

$$C_m = f_4(\alpha, Re, M)$$

$$C_m = \frac{M_A}{q_\infty S_{ref} D_{ref}}$$



$$F_{Ax} = C_A q_\infty S_{ref}$$

$$F_{Ay} = C_s q_\infty S_{ref}$$

$$F_{Az} = C_N q_\infty S_{ref}$$

$$M_{Ax} = C_l q_\infty S_{ref} D_{ref}$$

$$M_{Ay} = C_m q_\infty S_{ref} D_{ref}$$

$$M_{Az} = C_n q_\infty S_{ref} D_{ref}$$

$$C_A, C_S, C_N, C_L, C_m, C_n$$

$$f(\alpha, \beta, \dot{\alpha}, \dot{\beta}, p, q, r)$$

$$C_0, \quad \theta = A, S, N, l, m, n$$

for low angle of Attack,

$$C_0 = C_{0_0} + C_{0_\alpha} \alpha + C_{0_\beta} \beta + C_{0_{\dot{\alpha}}} \dot{\alpha} + \frac{D_{ref}}{2V_\infty} C_{0_{\dot{\alpha}}} \dot{\alpha} \\ + \frac{D_{ref}}{2V_\infty} C_{0_{\dot{\beta}}} \dot{\beta} + \frac{D_{ref}}{2V_\infty} C_{0_p} p + \frac{D_{ref}}{2V_\infty} C_{0_q} q + \frac{D_{ref}}{2V_\infty} C_{0_r} r$$

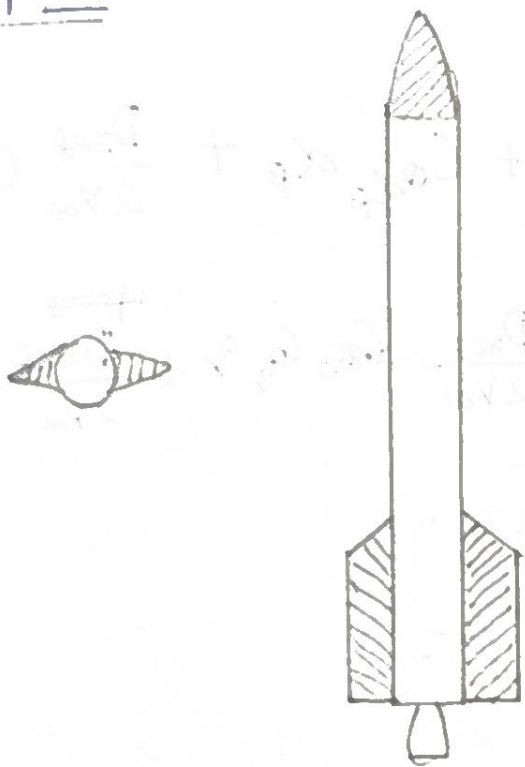
C_{0_0} = Coefficient at zero angle of Attack,

$$C_{0_\alpha} = \frac{\partial C_0}{\partial \alpha}$$

Date :- 27/03/2026

LV - in general Symmetric but functional requirements

Pitch plan



Pressure distribution
will be different
at these plane.

Yaw plane

In general :-

f_{Ax} , $f_{Ay} \approx 0$. at zero flight Angle of Attack

$C_{Axial} \rightarrow$ at $\alpha = 0 \rightarrow$ nearly same for $1^\circ - 2^\circ$ AoA.

Protrusions, Straps misalignments, fin misalignment $\rightarrow$ Rolling moment

$$f_{Ax} = -C_{Ax} q_\infty S_{ref}$$

C_{Ax} Axial force coefficient at 0° AoA.

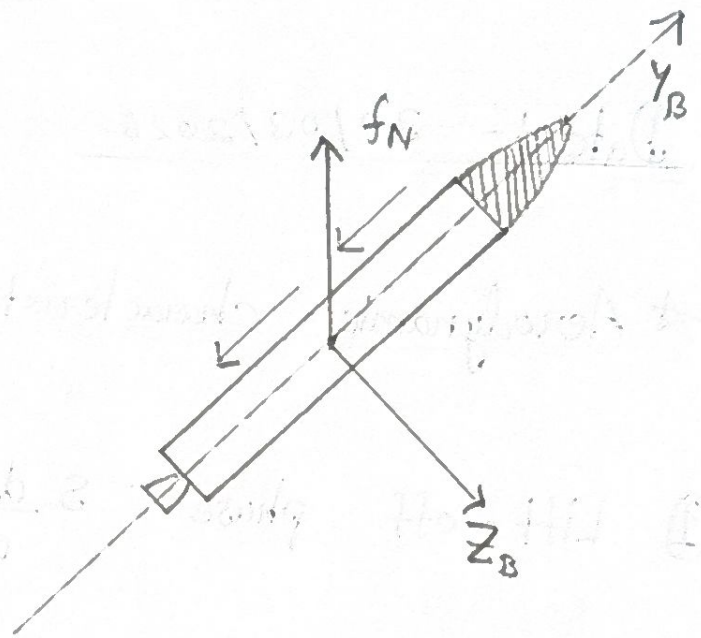
$$F_{Ay} = C_{sp} \beta q_{\infty} S_{ref}$$

$$F_{Az} = -C_{N\alpha} \alpha q_{\infty} S_{ref}$$

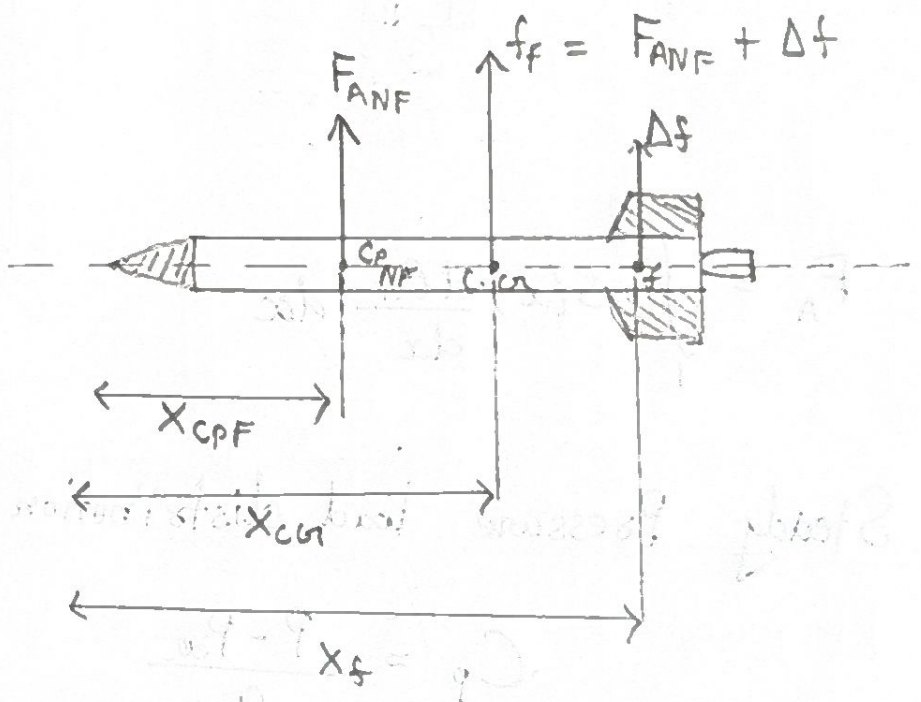
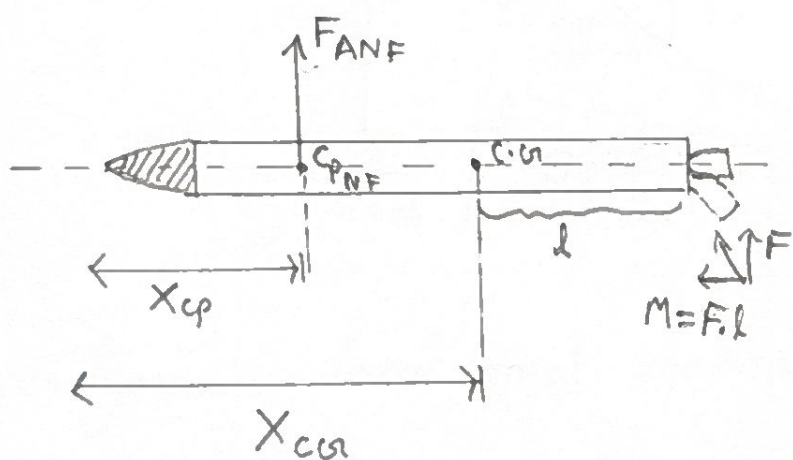
$$\rightarrow M_{Ax} = (C_{x,0} + C_{x,d_p} \alpha \beta + C_{x,s} S) q_{\infty} S_{ref} D_{ref}$$

$$M_{Ay} = C_{N\alpha} \alpha q_{\infty} S_{ref} (X_{CG} - X_{cp})$$

$$M_{Az} = C_{sp} \beta q_{\infty} S_x (X_{CG} - X_{cp})$$



to perform the simulation, we require angle, C.G. Variation data and all the coefficients



If in the socket the mach number increases and becomes Mach 1.1 then C_p and C_{cg} can remain the same.

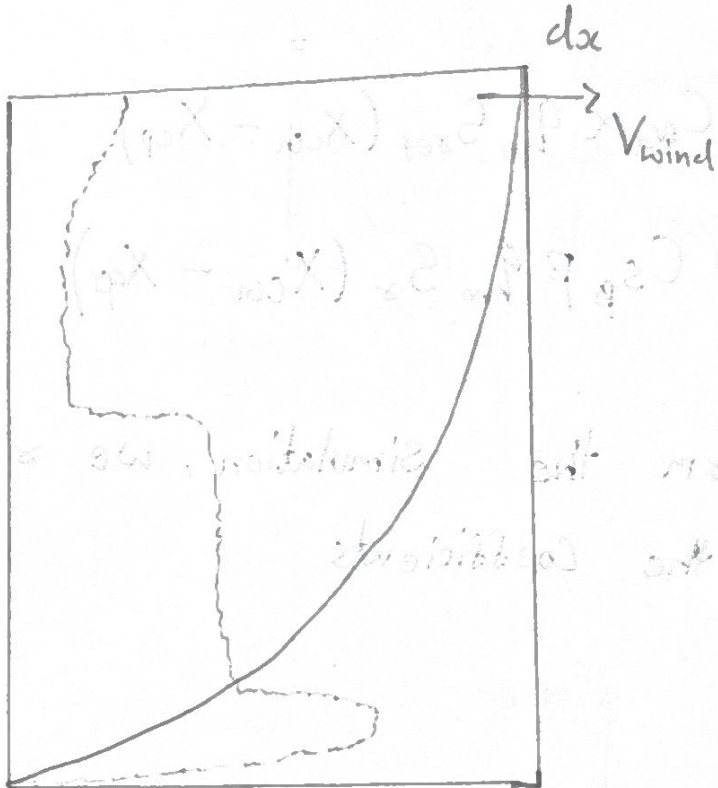
$$f_A = f_{ANF} + \Delta f$$

$$\{X_{cp}\}_f = \frac{f_{ANF} X_{cpNF} + \Delta f \times f}{f_{ANF} + \Delta f}$$

Date :- 30/03/2026

★ Aerodynamic characteristics and loads :-

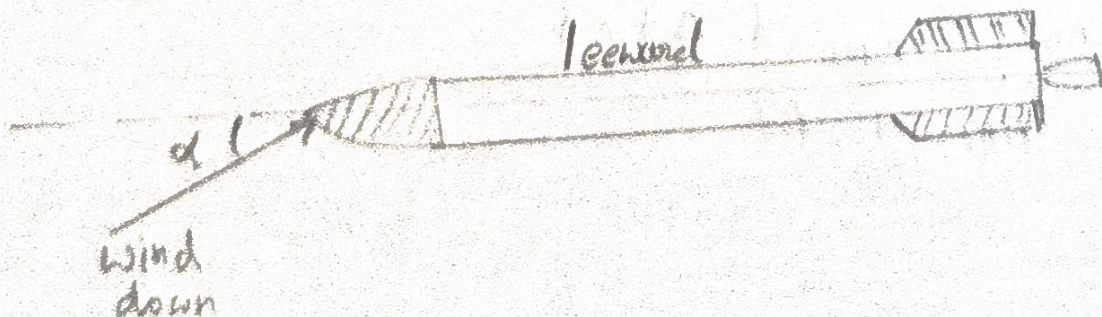
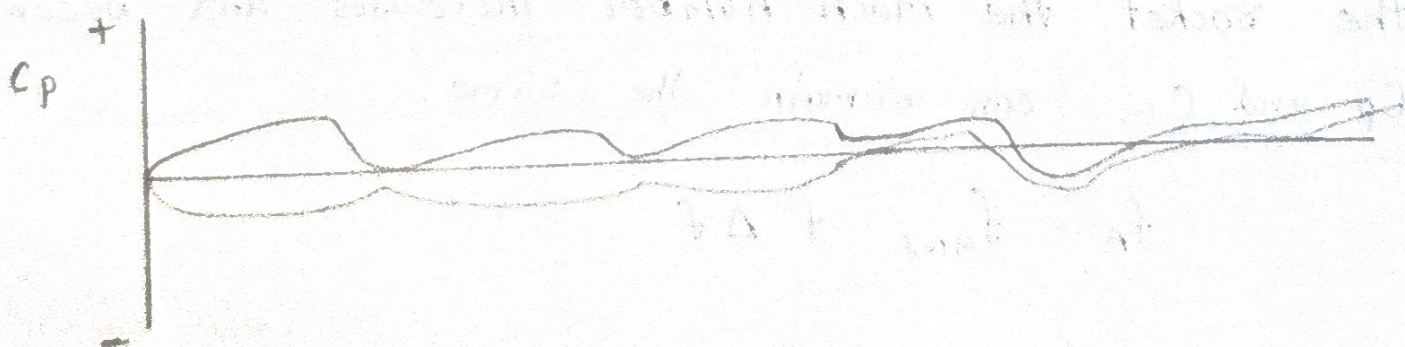
① Lift-off phase $S \frac{dC_N}{dx}$



$$F_A = \int_0^L S(x) \frac{dC_N}{dx} dx$$

Steady Pressure load distribution

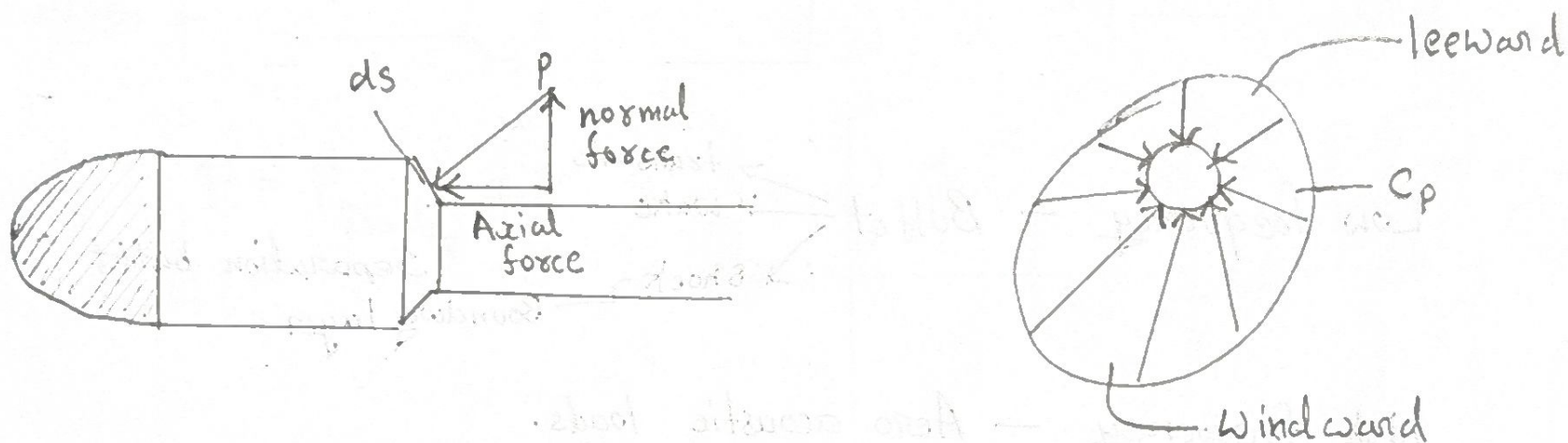
$$C_p = \frac{P - P_\infty}{q_\infty}$$



C_p^{+ve} — local pressure higher than atm pressure, P_o

C_p^{-ve} — local pressure lower

★ Cross Section C_p distribution



$$\rightarrow C_{N\alpha} = \frac{1}{S_{ref}} \int_0^l s \frac{dC_{N\alpha}}{dx} \cdot dx$$

$$C_A = \frac{1}{S_{ref}} \int_0^l s \frac{dC_A}{dx} \cdot dx$$

Aerodynamic force and moment $\rightarrow$ mission design

* Load distribution, $\frac{dC_{N\alpha}}{dx}$, $\frac{dC_A}{dx}$

① Sectional bending moments

② Shear force

③ Axial force

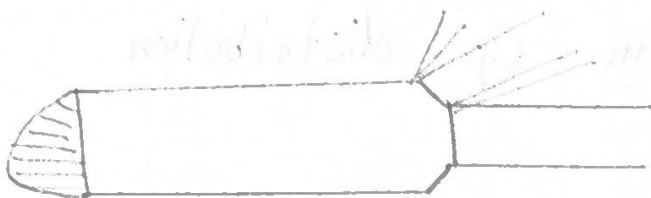
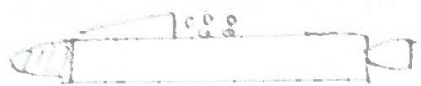
Rocket design should be done by considering structure, manufacturing, and materials $\rightarrow$ 3g acceleration

The Shape should be such that aerodynamic effects due to angle of attack are minimized.

★ Unsteady loads,



Separation

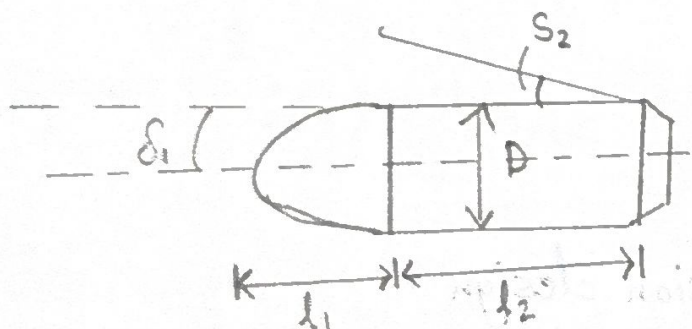


geometric
flow Sep.
+ reattachment

Low frequency - Buffet $\begin{cases} \text{transonic} \\ \text{wake} \\ \text{shock} \end{cases}$ $\begin{cases} \text{Separation buffet} \\ \text{Boundary layer} \end{cases}$

high frequency - Aero acoustic loads.

★ forebody Config - Critical NASA Guidelines



LV Config $\rightarrow$ design - Stable buffet Config $\left\{ \begin{array}{l} \text{accomodate} \\ \text{bigger payload} \end{array} \right.$

★ dynamic BM

★ aeroelastic wind tunnel test

★ flexible mode frequency.

$$f_{\text{model}} = \frac{f_{\text{vehicle}}}{L} \quad \left\{ L - \text{Scale down such} \right\}$$

$$BM(\text{Vehicle}) = \left(\frac{BM_{\text{vehicle}}}{\rho R L^2 X_{\text{model}}} \right) \times X_{\text{vehicle}}$$

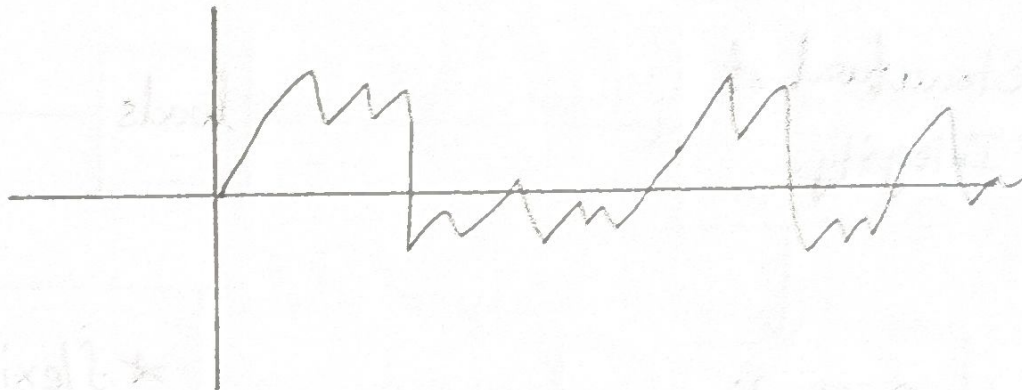
Q_R = dynamic pressure ratio

$X_{\text{model}}, X_{\text{vehicle}} \rightarrow$ Corresponding Specific location

★ Aero-acoustics

{ Impact of high frequency }

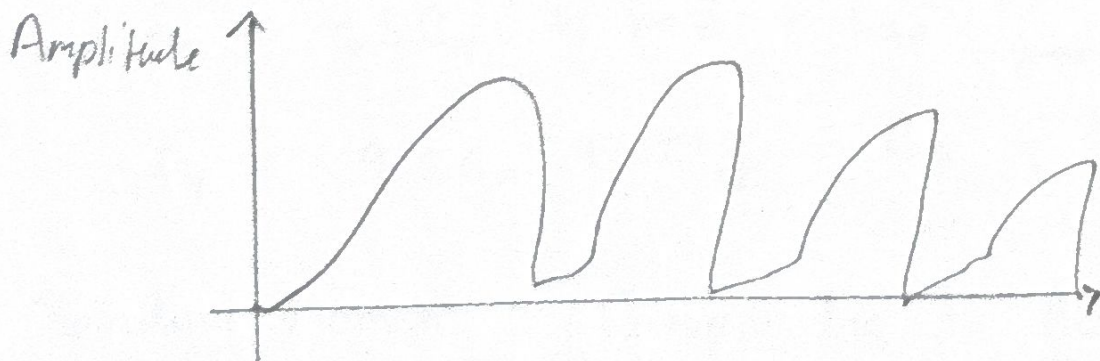
$$P = P_{\text{steady}} + P'$$



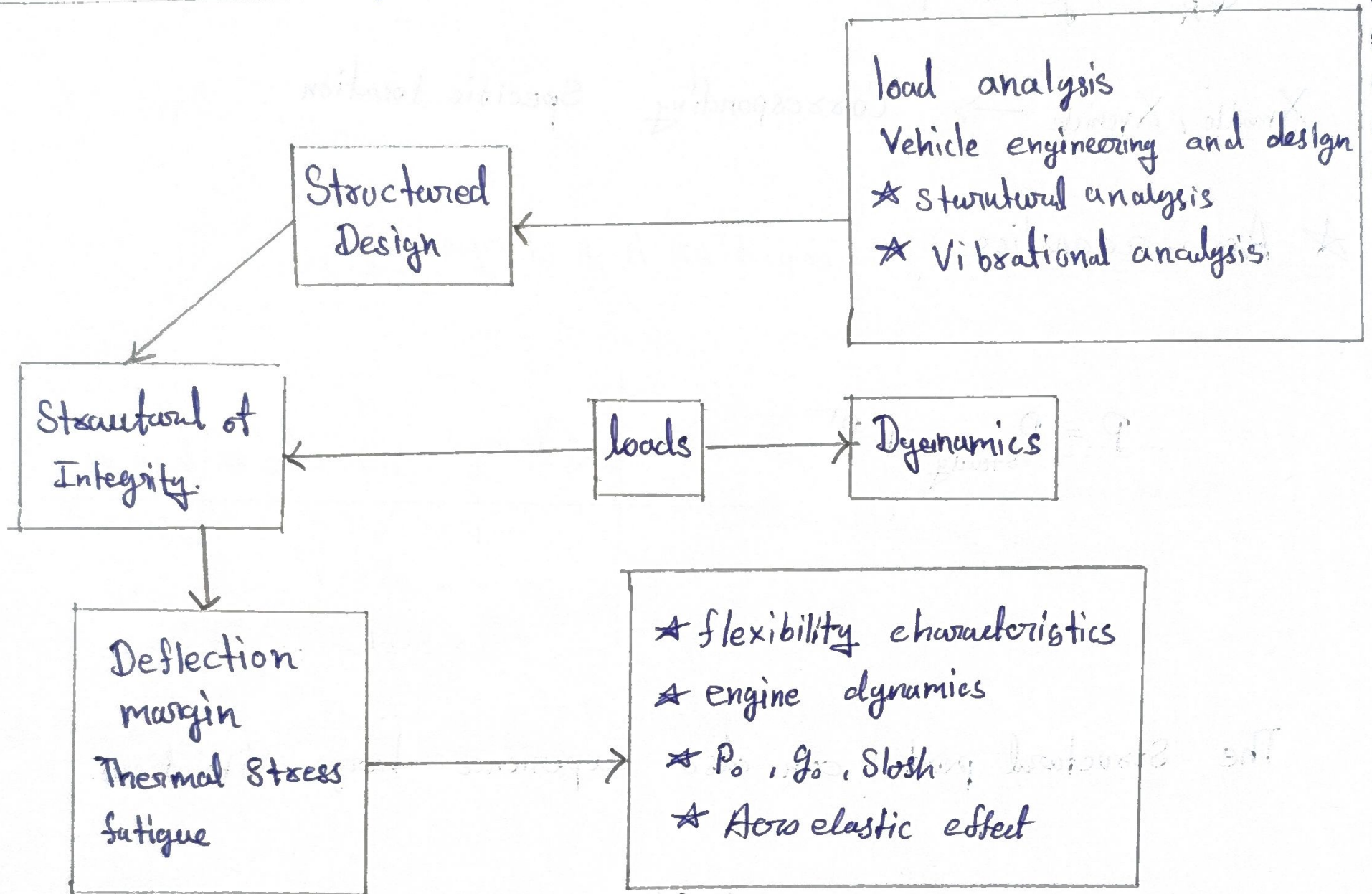
The Structural part can also experience large Vibrations.

$$\text{OASPL} = 20 \log_{10} \left(\frac{C_{p,\text{rms}}' \times q_{\infty}}{P_{\text{ref}}} \right)$$

$$\text{SPL} = 10 \log_{10} \left(\frac{\text{PSD}_{\text{integer}} \times \Delta f}{P_{\text{ref}}^2} \right)$$



★ Structural Design and analysis :-



Structures :-

Primary Structure & Secondary Structure

Structural loads :-

- ① ~~Stead~~ Steady loads
- ② load due to Bending moment
- ③ Axial forces on Complete Vehicle

Transient loads:-

- ① Thrust transient
- ② Control force
- ③ Acoustics

Thermal loads:-

- ① Aerodynamic heating
- ② Heat due to Combustion

★ Loads:-

- ① Axial loads

$$f_x = qSC_D + m\ddot{x}$$

- ② Shear force

$$f_z = qSC_N + m\ddot{z} + C_{sp}$$

$$f_y = qSC_{\pi} + m\ddot{y} + C_{f_{yaw}}$$

- ③ Bending Moment: Caused due to Aerodynamics, Control force, Buffet.

$$f_{x_b} = 2 \times \frac{\text{Total Bending Moment}}{\text{Radius of the Structure}}$$

Total equivalent axial load (EAL)

$$F_{x_c} = f_x + f_{x_b}$$

Initial
Design phase

Aerodynamic - give 20% dispersion

Bending moment:- 40-60% dispersion

$$\text{factor of Stability (fs)} = \frac{\text{Allowable load}}{\text{Design load}}$$

related to yield strength or ultimate strength or buckling length.

$$\text{Margin of Safety (M.S)} = fs - 1$$

(MS $\rightarrow$ Slightly greater than zero)

★ limit load $\rightarrow$ max. expected load on structures.

★ Load factor $\rightarrow$ Multiplication factor applied on limit load

★ proof load $\rightarrow$ does not cause global weights, ultimate load $\rightarrow$ " " " acmg failure.

★ Unmanned flight :-

Proof load - 1.11 - 1.15 times the limit load

Ultimate load - 1.4 - 2.0 times limit load

★ Motor Case :-

$$t_s = \frac{P_d}{2\sigma} (1 + m_s)$$

thickness

★ Inter stage Structure :-

★ failure mode - buckling

★ Average Stress deductions

Total Compression load $f_{xe} = f_x + \frac{4M}{d}$

Average Stress $= \frac{f_{xe}}{A}$

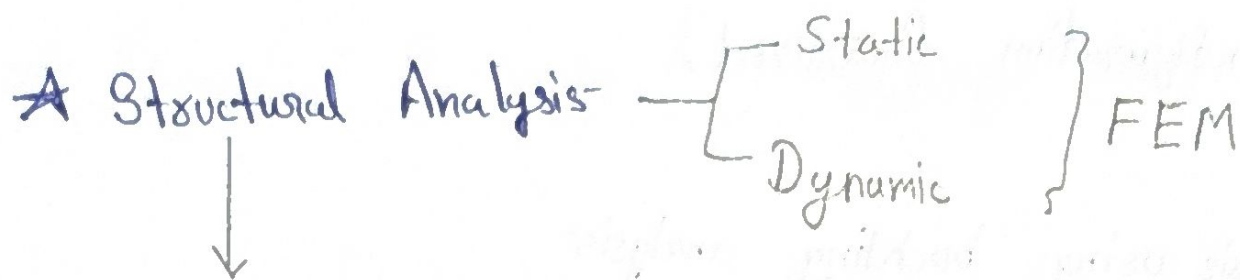
★ Structural joints-

$$f_{mass} = \frac{4M}{dn}$$

mass load on bolts

n = number of bolts

d = bolt pitch circle diameter



- ① Static analysis - estimate stress and strain
- ② Dynamic analysis - estimate frequency, mode shapes, dynamic response
- ③ Stability analysis - buckling analysis
- ④ Interdisciplinary Coupled analysis - Aero elastic, thermo-Structural

★ Static Analysis (Stress/Strain Analysis)

* fundamental equation in FEM

$$\{F\} = [K]\{\delta\}$$

$\{f\}$ = applied force motion (N)

$[K]$ = Global Stiffness matrix (N/m)

$\{\delta\}$ = unknown displacement

$[K]$ = elemental Stiffness matrix $\rightarrow$ evaluated using minimum potential Energy

Once $\{\delta\}$ is estimated

$$\{\epsilon\} = [B]\{\delta\}$$

$$\{\sigma\} = [D][\epsilon]$$

$[D]$ = material Constant
depends on plane motion,
Stress, plane strain, bending, Composite Construction etc.

$\{\sigma\}$ - Stress Component matrix

$\{\epsilon\}$ - Strain matrix

$[B]$ - differential matrix

$\{\delta\}$ - displacement matrix

★ Once Structural Configuration Finalized,

↓
Static Stability, analysis using buckling analysis

↓
Estimate Eigen Values

★ Structural dynamic Analysis

Launch Vehicle $\rightarrow \frac{L}{D} \uparrow \rightarrow$ flexible Structure

① Dynamic characteristics
free vibration frequencies, mode shapes etc.

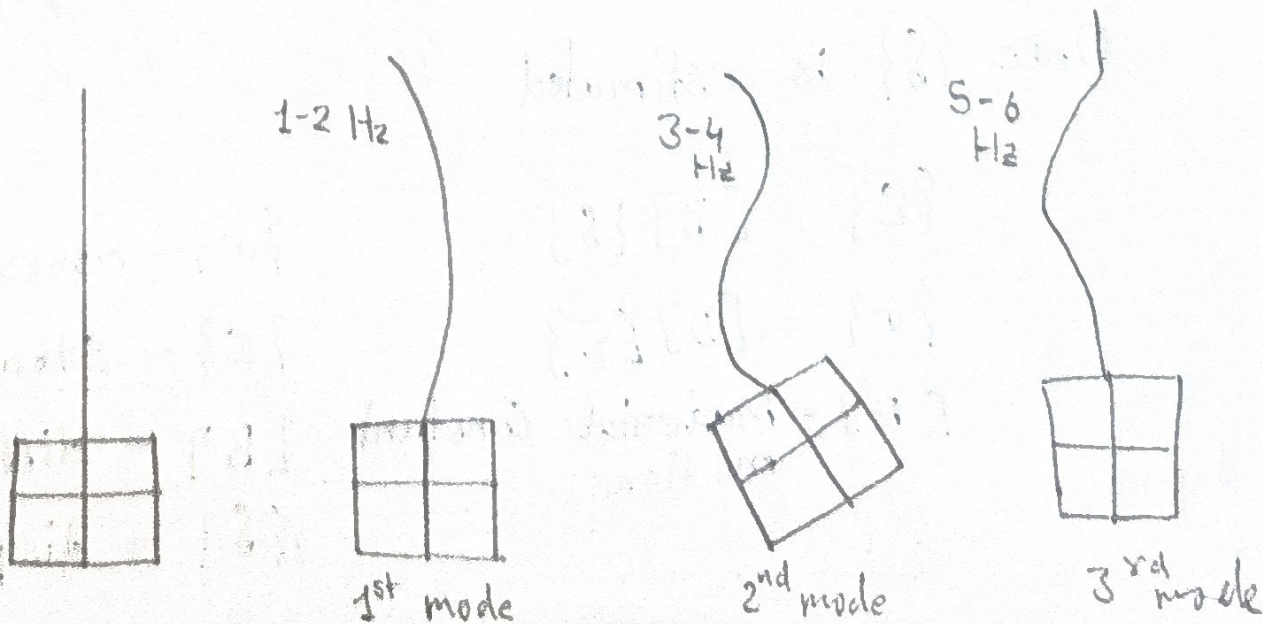
② Dynamic excitations-

③ Dynamic response and Stability,

- Coupled load analysis
- Pogo (resonance due to fluid flow in pipe structure (in liquid engine))
- Control-Structure interaction
- Aero-elastic (low Buffet freq. & high freq.)

$$M \{\ddot{x}\} + K \{x\} = 0$$

* beam model



★ Dynamic Response & Stability analysis

$$[M] \{\ddot{x}\} + [C] \{\dot{x}\} + [K] \{x\} = \{f\}$$

★ Response analysis Carried out in time domain

★ Stability analysis $\rightarrow$ S domain

★ for linear Systems

$$\{x\} = \sum_{i=1}^n \Phi_i q_i \quad \text{--- (1)}$$

Φ_i - normalized mode shape

q_i - generalized Coordinates of the i^{th} mode

$$\{\ddot{q}\} + 2[\zeta][\omega][\dot{q}] + [\omega^2][q] = \frac{\{\Phi\}^T \{f\}}{\{\Phi\}^T [M] \{\Phi\}} \quad \text{--- (2)}$$

$[\zeta]$ = mode damping ratio

$[\omega]$ = mode shape frequency

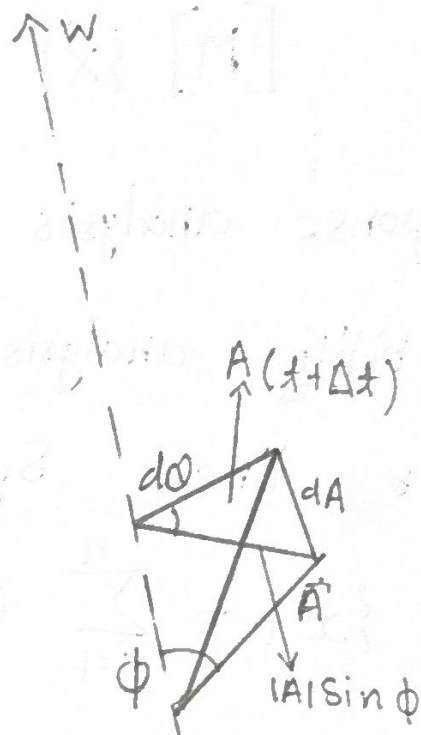
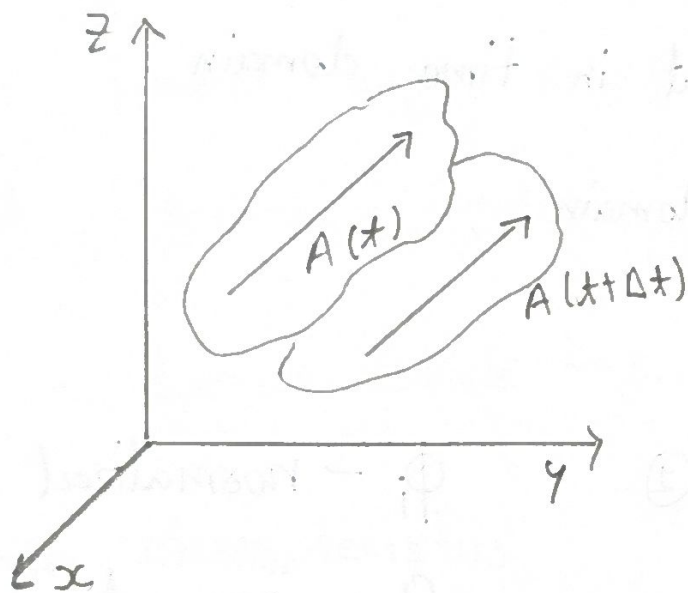
★ Eqⁿ (1) & eqⁿ (2) Solved together for getting displacement, velocity, acceleration.

★ for Stability analysis $\rightarrow$ S domain $\rightarrow$ laplace transform
 $\downarrow$
complex design

Chapter-7 :- Rigid Body Dynamics

Ref - Curtis
Sch-13

Time Derivative of moving Vector,



$$\frac{d\vec{A}}{dt} \left\{ \begin{array}{l} \frac{d\vec{A}}{dt} = (|\vec{A}| \sin \phi |\vec{\omega}|) dt \hat{n} \\ d\vec{A} = (|\vec{\omega}| |\vec{A}| \sin \phi \hat{n}) dt \end{array} \right. \quad d\vec{A} = (\vec{\omega} \times \vec{A}) dt$$

$$\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A} \quad \text{if} \quad \frac{d}{dt} |\vec{A}| = 0$$

$$\vec{A}(t+\Delta t) = \vec{A}(t) + d\vec{A}$$

$$|d\vec{A}| = |\vec{A}| \sin \phi d\theta$$

$\downarrow$
 $|\vec{\omega}| dt$

$$\alpha = \frac{d\vec{\omega}}{dt}, \quad \frac{d^2\vec{A}}{dt^2} = \frac{d}{dt} \frac{d\vec{A}}{dt} = \frac{d}{dt} (\vec{\omega} \times \vec{A})$$

$$= \frac{d}{dt} \vec{\omega} \times \vec{A} + \vec{\omega} \times \frac{d\vec{A}}{dt}$$

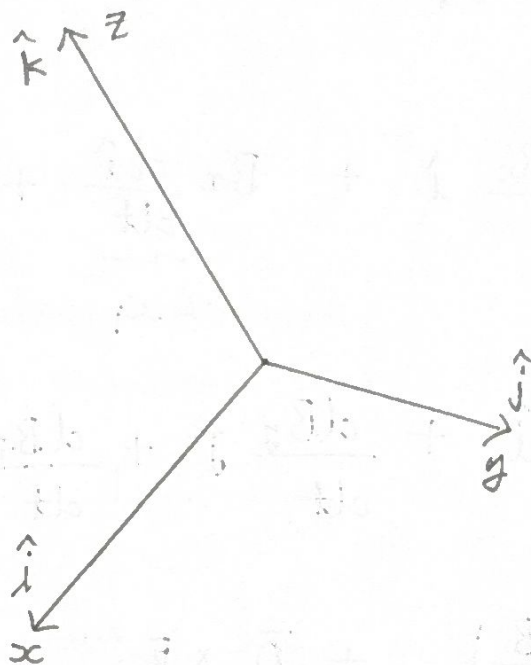
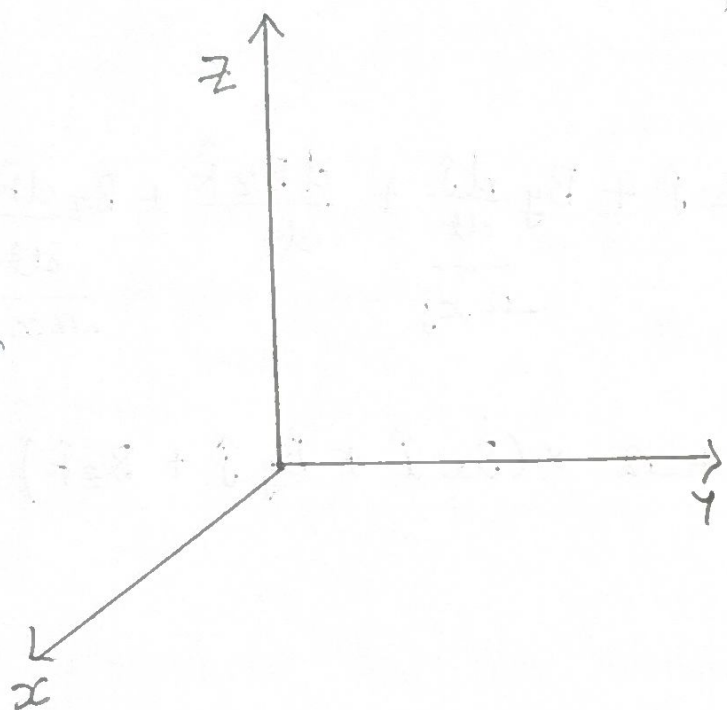
$$= \vec{\alpha} \times \vec{A} + \vec{\omega} \times (\vec{\omega} \times \vec{A})$$

$$\frac{\vec{\alpha} \times \vec{r}}{1}$$

lies on rotating plane

$$\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

★ Fixed and Moving rigid frame of Reference :-



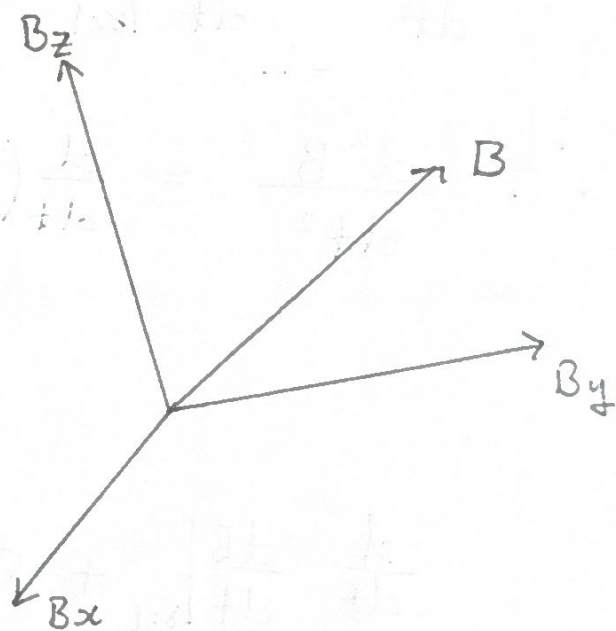
XYZ - rigid inertial frame

xyz - rigid moving frame of reference

→ translating or rotating Body.

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\vec{B}' = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$



★ Motion of moving frame :- (x, y, z)

Absolute angular Velocity Ω

not only translation but also rotates

Body frames → axes → body axes

Consider,

$$\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\frac{d\vec{B}}{dt} = \frac{dB_x}{dt} \hat{i} + \frac{dB_y}{dt} \hat{j} + \frac{dB_z}{dt} \hat{k}$$

$$\left. \frac{d\vec{B}}{dt} \right|_{\text{rel}} = \frac{dB_x}{dt} \hat{i} + \frac{dB_y}{dt} \hat{j} + \frac{dB_z}{dt} \hat{k}$$

$$\frac{d\vec{B}}{dt} = \frac{dB_x}{dt} \hat{i} + B_x \underbrace{\frac{d\hat{i}}{dt}}_{-\Omega \times \hat{i}} + \frac{dB_y}{dt} \hat{j} + B_y \underbrace{\frac{d\hat{j}}{dt}}_{-\Omega \times \hat{j}} + \frac{dB_z}{dt} \hat{k} + B_z \underbrace{\frac{d\hat{k}}{dt}}_{-\Omega \times \hat{k}}$$

$$\frac{d\vec{B}}{dt} = \frac{dB_x}{dt} \hat{i} + \frac{dB_y}{dt} \hat{j} + \frac{dB_z}{dt} \hat{k} + \Omega \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$\therefore \frac{d\vec{B}}{dt} = \left. \frac{d\vec{B}}{dt} \right|_{\text{rel}} + \Omega \times \vec{B}$$

$$\therefore \frac{d^2 \vec{B}}{dt^2} = \frac{d}{dt} \left(\left. \frac{d\vec{B}}{dt} \right|_{\text{rel}} + \Omega \times \vec{B} \right)$$

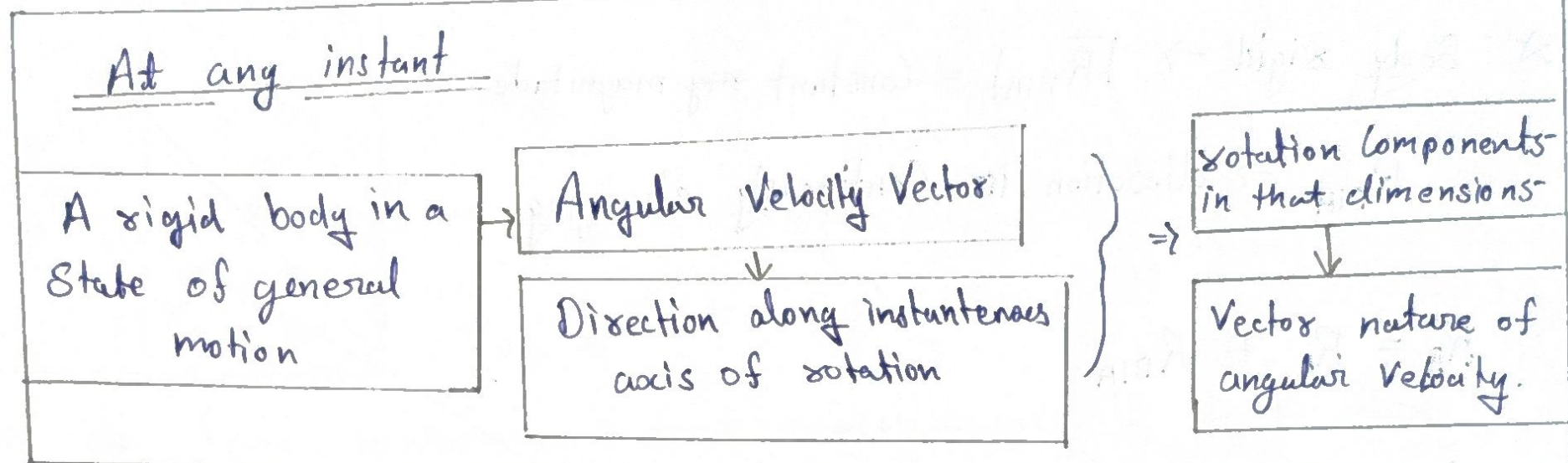
$$= \underbrace{\frac{d}{dt} \left(\left. \frac{d\vec{B}}{dt} \right|_{\text{rel}} \right)}_A + \frac{d}{dt} (\Omega \times \vec{B})$$

$$\frac{d}{dt} \left. \frac{d\vec{B}}{dt} \right|_{\text{rel}} + \Omega \times \left. \frac{d\vec{B}}{dt} \right|_{\text{rel}}$$

$$\frac{d}{dt} \Omega \times \vec{B} + \Omega \times \left(\left. \frac{d\vec{B}}{dt} \right|_{\text{rel}} + \Omega \times \vec{B} \right)$$

$$\frac{d^2 \vec{B}}{dt^2} = \left. \frac{d^2 \vec{B}}{dt^2} \right|_{\text{rel}} + \dot{\Omega} \times \vec{B} + \Omega \times (\Omega \times \vec{B}) + 2\Omega \times \left. \frac{d\vec{B}}{dt} \right|_{\text{rel}}$$

Date :- 08/04/2026



★ Interaction b/w the motion of a rigid body and the forces acting on it.

★ Translational Component (motion) $\Rightarrow$ ★ Centre of mass
★ Apply particle mechanics to determine motion

★ Rotational dynamics (motion) $\Rightarrow$ requires body angular momentum.

↓
Also requires info on mass distribution (I)

↓
described by six component of moment of inertia tensor

→ Specify the orientation of a body in 3D Space.

→ Euler angles

→ Roll, pitch, Yaw (Suitable for airplane / L.V.)

→ Rates $\Rightarrow$ Quaternions (Avoid Singularity)

★ $\vec{R}_{B/A}$ - position Vector of B relative to A

★ Body rigid $\rightarrow |\vec{R}_{B/A}| = \text{Constant}$ magnitude

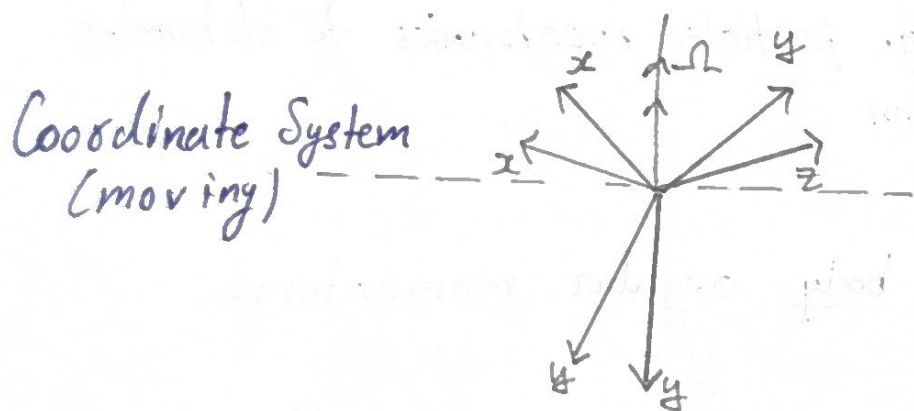
$\vec{R}_{B/A} \rightarrow$ direction is continuously changing.

$$\vec{R}_B = \vec{R}_A + \vec{R}_{B/A}$$

$$\dot{\vec{R}}_B = \dot{\vec{R}}_A + \underbrace{\frac{d}{dt} \vec{R}_{B/A}}_{\vec{\omega} \times \vec{R}_{B/A}} \rightarrow \text{time derivative} \rightarrow$$

$$\boxed{\vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{R}_{B/A}}$$

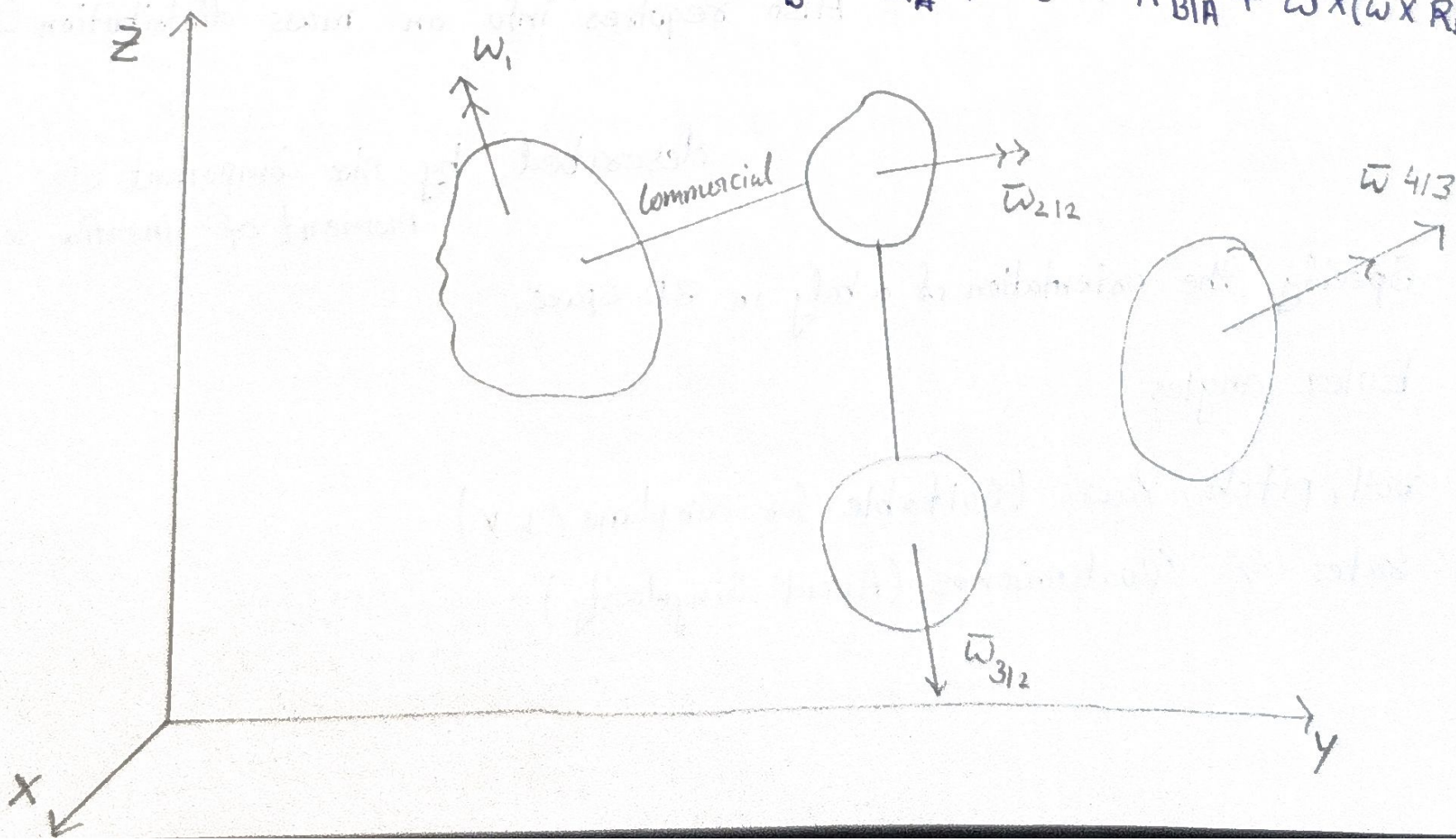
$$\frac{d}{dt} \dot{\vec{R}}_B = \frac{d}{dt} \dot{\vec{R}}_A + \frac{d}{dt} \left(\frac{d}{dt} \vec{R}_{B/A} \right)$$



$$\vec{a}_B = \vec{a}_A + \frac{d}{dt} (\vec{\omega} \times \vec{R}_{B/A})$$

$$\vec{a}_B = \vec{a}_A + \underbrace{\frac{d}{dt} \vec{\omega}}_{\alpha} \times \vec{R}_{B/A} + \vec{\omega} \times \frac{d\vec{R}_{B/A}}{dt}$$

$$\vec{a}_B = \vec{a}_A + \alpha \times \vec{R}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{R}_{B/A})$$



$$\bar{\omega}_4 = \bar{\omega}_{4/3} + \bar{\omega}_{3/2} + \bar{\omega}_{2/1} + \bar{\omega}_1$$

$$\bar{\omega} = \bar{\omega}_1 + \bar{\omega}_{2/1} + \bar{\omega}_{3/2} + \bar{\omega}_{4/3}$$

$$\bar{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$$\bar{\omega} = \omega_{x'} \hat{i}' + \omega_{y'} \hat{j}' + \omega_{z'} \hat{k}'$$

if we have information about $\omega_x(t)$, $\omega_y(t)$ & $\omega_z(t)$ then

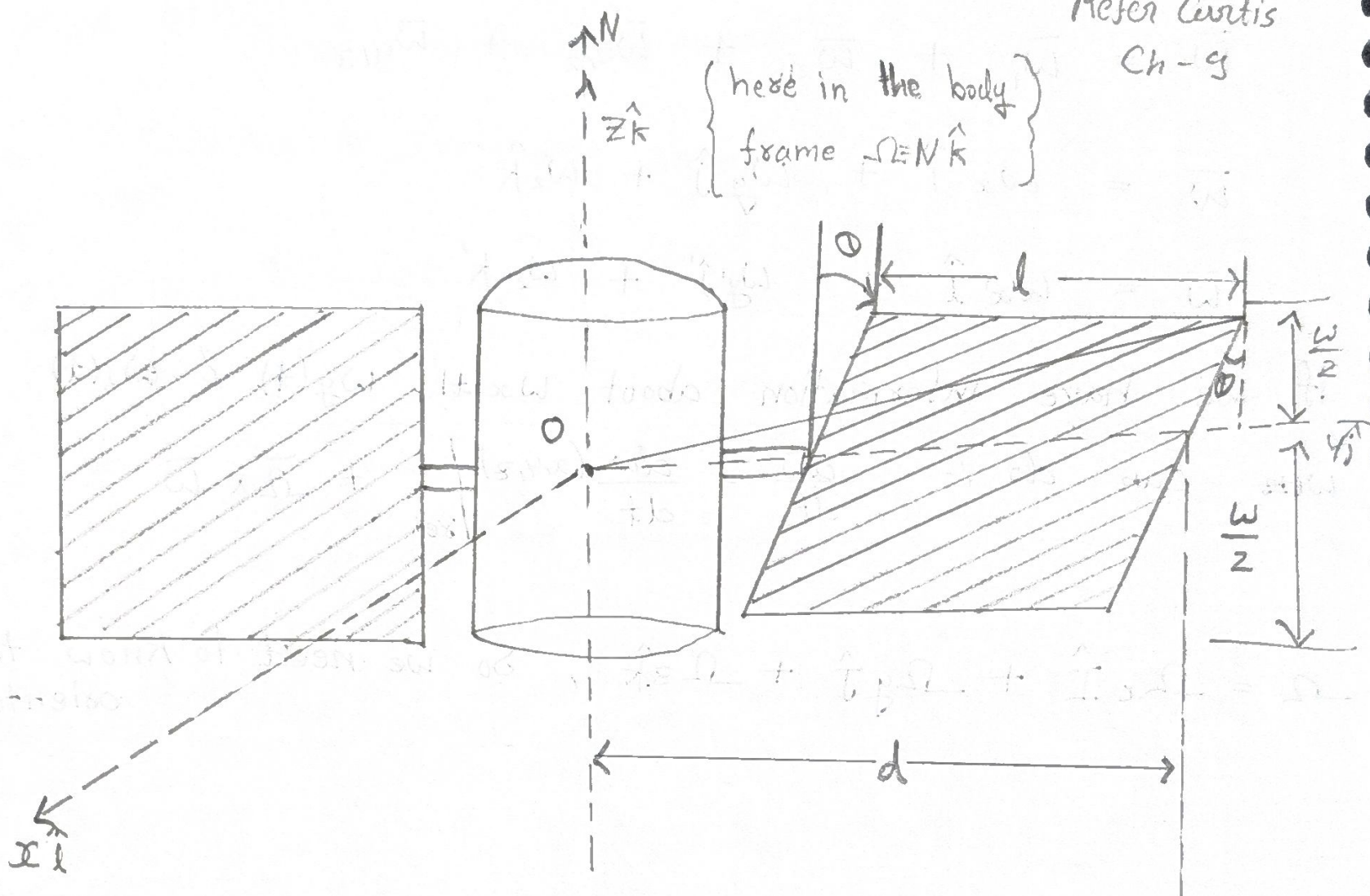
we can do :- $\frac{d\bar{\omega}}{dt} = \frac{d\bar{\omega}}{dt} \Big|_{rel} + \bar{\Omega} \times \bar{\omega}$

$\bar{\Omega} = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$, So we need to know the orientation

Date :- 10/04/2026

Rotating Star panel of a rotating Satellite

Refer Curtis
Ch - 9



★ Satellite rotating about z -axis at constant rate ' N '

★ xyz axes attached to the spacecraft.

★ z -axis $\rightarrow$ fixed orientation in inertial space.

★ Solar panels rotate at const. rate $\dot{\theta}$ as shown.

Qn

★ Relative to point 'O', Calculate for point A.

(a) $V_{A/O}$

(b) $a_{A/O}$

$$\rightarrow \Omega = N \hat{k}$$

★ absolute angular velocity of the panel.

= Absolute angular Velocity of the Spacecraft

+

Angular Velocity of the panel relative to the Spacecraft.

$$\omega_{\text{panel}} = \dot{\phi} \hat{j} + N \hat{k}$$

$$\vec{r}_A = \vec{r}_O + \vec{r}_{A/O}$$

line joining from O to A Vector with respect to some fixed frame.

$$\therefore \vec{r}_A - \vec{r}_O = \vec{r}_{A/O}$$

$$\boxed{\frac{d\vec{r}_A}{dt} = \frac{d\vec{r}_O}{dt} = \frac{d}{dt} \vec{r}_{A/O}} \quad \text{--- ①}$$

I have information only with respect to Satellite (moving) frame

So, $\vec{\omega} \times \vec{r}_{A/O} = \frac{d}{dt} \vec{r}_{A/O}$. We should not use but it can work.

instead we can use ① as equation.

$$\vec{\omega} \times \vec{r}_{A/O} \rightarrow (x, y, z) \rightarrow i, j, k.$$

$$\omega_{\text{panel}} = -\dot{\phi} \hat{j} + N \hat{k}$$

$$\vec{r}_{A/O} = \frac{-\omega}{2} \sin \phi \hat{i} + d \hat{j} + \frac{\omega}{2} \cos \phi \hat{k}$$

So,

$$\bar{\omega} \times \bar{r}_{A10} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -\dot{\theta} & N \\ -\frac{\omega}{2} \sin \theta & d & \frac{\omega}{2} \cos \theta \end{vmatrix}$$

$$V_{A10} = -\left(\frac{\omega}{2} \dot{\theta} \cos \theta + Nd\right) \hat{i} - \frac{\omega}{2} N \sin \theta \hat{j} - \frac{\omega}{2} \dot{\theta} \sin \theta \hat{k}$$

$$\frac{d}{dt} \left(\frac{d r_a}{dt} \right) = \frac{d}{dt} r_o$$

$$\therefore a_a - a_o = \frac{d}{dt} \frac{d}{dt} (r_{A10}) = \frac{d \bar{\omega}}{dt} \times r_{A10} + \bar{\omega} \times \frac{d}{dt} (r_{A10})$$

$$\frac{d}{dt} (\bar{\omega} \times r_{A10}) \quad \alpha_{\text{panel}} \times r_{A10} + \bar{\omega} \times (\bar{\omega} \times r_{A10})$$

$$\alpha_{\text{panel}} = \frac{d}{dt} (\omega_{\text{panel}})$$

$$= -\dot{\theta} \hat{j} + N \hat{k}$$

$$\omega_{\text{panel}} = A \hat{i} + B \hat{j} + C \hat{k}$$

$$N \hat{i} \times (-\dot{\theta} \hat{j} + N \hat{k}) = N \dot{\theta} \hat{i}$$

$$\frac{d}{dt} (\omega_{\text{panel}}) = \frac{d}{dt} \omega_{\text{pm}} \Big|_{\text{rel}} + \underbrace{\Omega \times \omega_{\text{p/m}}}_{=0}$$

$$\alpha = N \dot{\theta} \hat{i}$$

$$\therefore a_{A10} = \alpha_{\text{panel}} \times r_{A10} + \bar{\omega} \times (\bar{\omega} \times r_{A10})$$

74



- * xyz axes are attached to Gimbal ring $\Rightarrow$ where angle θ with vertical is increasing. at a const. $\dot{\alpha} = \dot{\theta}$

★ Assembly forced to recess at a constant rate N around the vertical

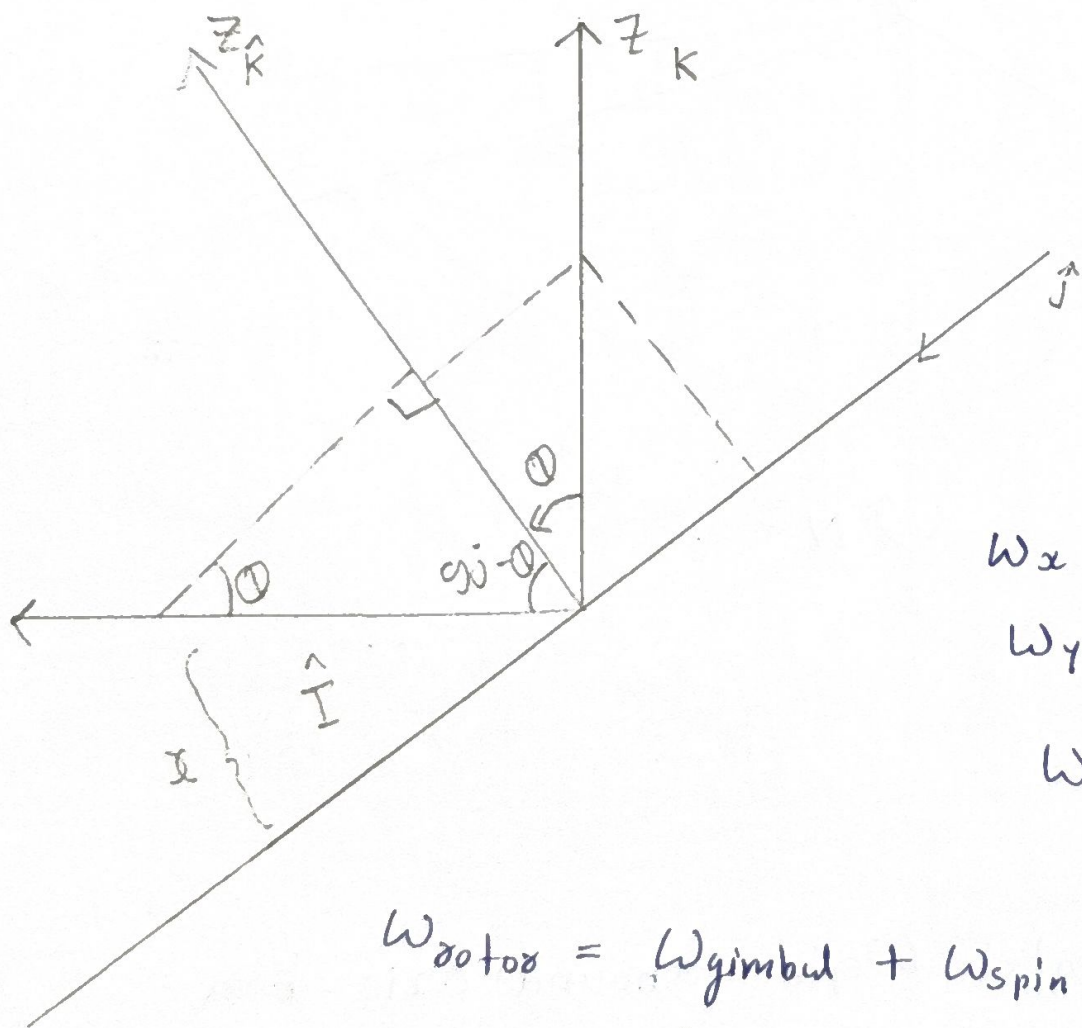
★ for the rotor position shown.

find,

- absolute angular velocity of the rotor
- " " acceleration of the rotor.
- express the results in both XYZ frame and xyz frame.

$$\rightarrow \text{gimbal ring} = \left[\begin{array}{c} N\hat{k} \text{ --- Capillary} \\ + \underbrace{\dot{\theta}\hat{i}}_{\text{small } i} \end{array} \right] = N(\sin\theta\hat{j} + \cos\theta\hat{k}) + \dot{\theta}\hat{i}$$

all are small



$$\hat{I} = -\cos\theta\hat{j} + \sin\theta\hat{k}$$

$$\hat{j} = \hat{i}$$

$$\hat{k} = \sin\theta\hat{j} + \cos\theta\hat{k}$$

$$\begin{aligned} \omega_x &= \begin{bmatrix} 0 & -\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \\ \omega_y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \\ \omega_z &= \begin{bmatrix} 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \end{aligned}$$

$$\omega_{\text{rotor}} = \omega_{\text{gimbal}} + \omega_{\text{spin}}\hat{k}$$

$$= \dot{\theta}\hat{i} + N\sin\theta\hat{j} + N\cos\theta\hat{k} + \omega_{\text{spin}}\hat{k}$$

$$= \dot{\theta}\hat{i} + N\sin\theta\hat{j} + (N\cos\theta + \omega_{\text{spin}})\hat{k}$$

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} 0 & -\cos\theta & \sin\theta \\ 1 & 0 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ N\sin\theta \\ N\cos\theta + \omega_{\text{spin}} \end{bmatrix}$$

$$\frac{d\omega_{\text{rotor}}}{dt} = \frac{d\omega}{dt} \bigg|_{\text{re}} + \Omega \times \omega_{\text{rotor}}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{\theta} & N\sin\theta & N\cos\theta \\ \ddot{\theta} & N\sin\theta & N\cos\theta + \omega_{\text{spin}} \end{vmatrix}$$

$$\alpha_{\text{rotor}} = N\omega_{\text{spin}}\sin\theta\hat{i} + \ddot{\theta}(N\cos\theta - \omega_{\text{spin}})\hat{j} - N\dot{\theta}\sin\theta\hat{k}$$

Date: 21/04/2026

★ Equation of translational motion

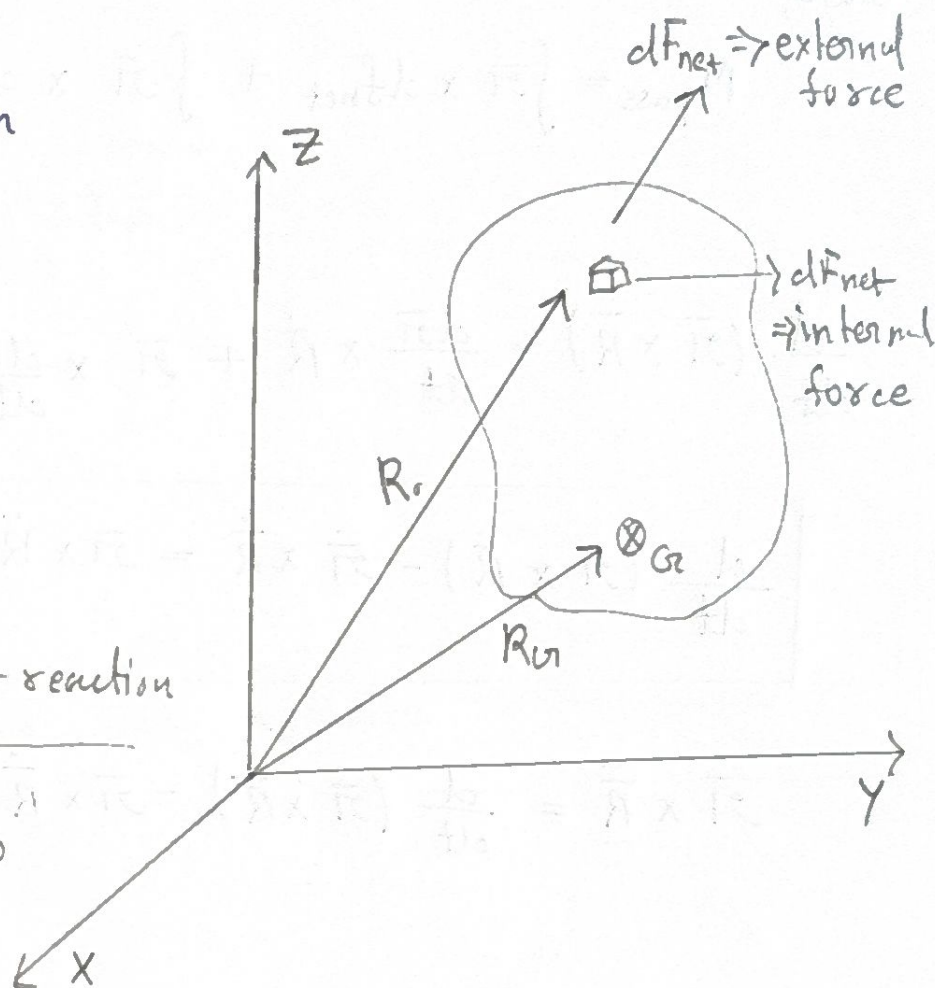
$$dF_{net} + df_{net} = dm \ddot{R}$$

$$\int dF_{net} + \int df_{net} = \int_m \ddot{R} dm$$

internal force will occur in action-reaction pairs

Integral will be zero

$$\therefore F_{net} = \int_m \ddot{R} dm = m \ddot{R}_G$$



★ Equation of Rotational motion:-

P - Orbital

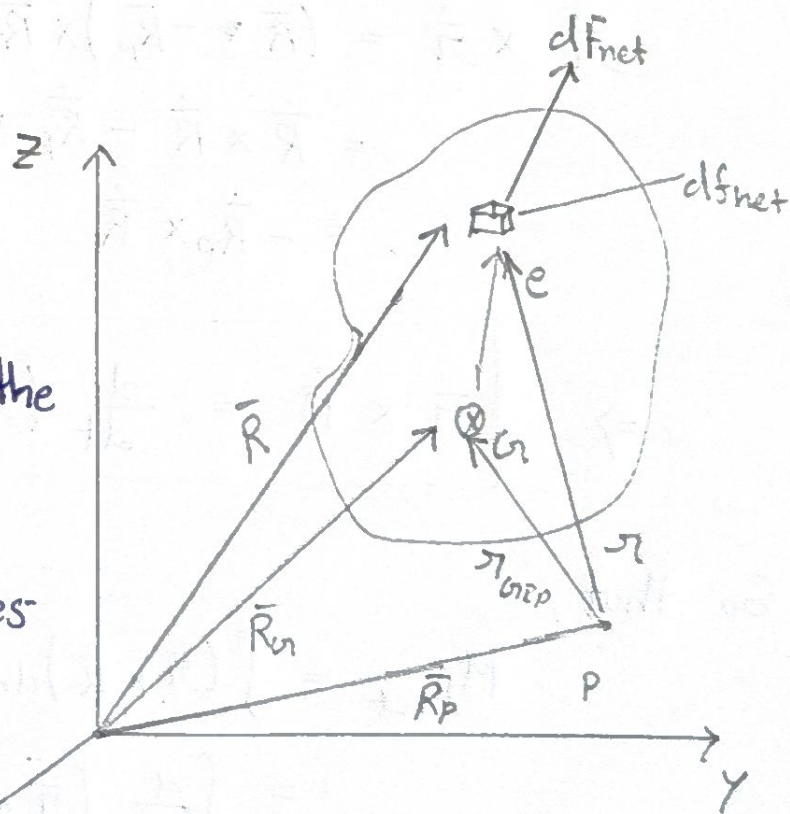
- need not be fixed in Space.
- need not be attached to any point in the body.

★ Moment about point P of the ~~stress~~ forces on mass element dm

$$dM_P = \vec{r} \times (d\vec{F}_{net} + d\vec{f}_{net})$$

$$\int dM_P = \vec{r} \times (dm \ddot{R})$$

$$M_{P_{net}} = \int \vec{r} \times (dm \ddot{R})$$



also,

$$M_{pass} = \int \vec{r} \times d\vec{f}_{net} + \int \vec{r} \times d\vec{f}_{net}$$

→ will go to zero because to action reaction
pairs and corresponding moment

$$\frac{d}{dt} (\vec{r} \times \vec{R}) = \frac{d\vec{r}}{dt} \times \vec{R} + \vec{r} \times \frac{d\vec{R}}{dt}$$

$$\boxed{\frac{d}{dt} (\vec{r} \times \vec{R}) - \vec{r} \times \dot{\vec{R}} = \vec{r} \times \ddot{\vec{R}}}$$

$$\vec{r} \times \ddot{\vec{R}} = \frac{d}{dt} (\vec{r} \times \dot{\vec{R}}) - \vec{r} \times \dot{\vec{R}}$$

$$\vec{r} = \vec{R} - \vec{R}_p$$

momentum will be with respect to $\vec{R}_p$

$$\begin{aligned} \vec{r} \times \dot{\vec{r}} &= (\vec{R} - \vec{R}_p) \times \dot{\vec{R}} \\ &= \vec{R} \times \dot{\vec{R}} - \vec{R}_p \times \dot{\vec{R}} \\ &= -\vec{R}_p \times \dot{\vec{R}} \end{aligned}$$

$$\Rightarrow \boxed{\vec{r} \times \dot{\vec{R}} = \frac{d}{dt} (\vec{r} \times \dot{\vec{R}}) + \vec{R}_p \times \dot{\vec{R}}}$$

So that,

$$\begin{aligned} M_{p_{net}} &= \int (\vec{r} \times \ddot{\vec{R}}) dm \\ &= \int \frac{d}{dt} (\vec{r} \times \dot{\vec{R}}) dm + (\vec{R}_p \times \dot{\vec{R}}) dm \\ &= \frac{d}{dt} \int (\vec{r} \times \dot{\vec{R}}) dm + \vec{R}_p \times \int \dot{\vec{R}} dm \end{aligned}$$

$$\vec{r} \times \dot{\vec{R}} dm$$

linear momentum of elemental mass dm



↓
moment of the linear momentum

↓
Angular momentum of the elemental body.

$$\frac{d}{dt} \int \vec{r} \times \dot{\vec{R}} dm$$

↓

$$\int \vec{r} \times \dot{\vec{R}} dm = \vec{H}_P$$

$\vec{H}_P$ is the angular momentum of the body relative to the point P.

$$M_{P_{\text{net}}} = \frac{d}{dt} \vec{H}_P + \dot{\vec{R}}_P \times \int \vec{R} dm$$

Reaction is $\neq$ Conservation of momentum, wheel.

$\vec{r} \times \dot{\vec{r}} dm$
 linear momentum
 of elemental mass dm
 $\Downarrow$
 moment of the linear momentum
 $\Downarrow$
 Angular momentum of the elemental body

$$\frac{d}{dt} \int \vec{r} \times \dot{\vec{r}} dm$$

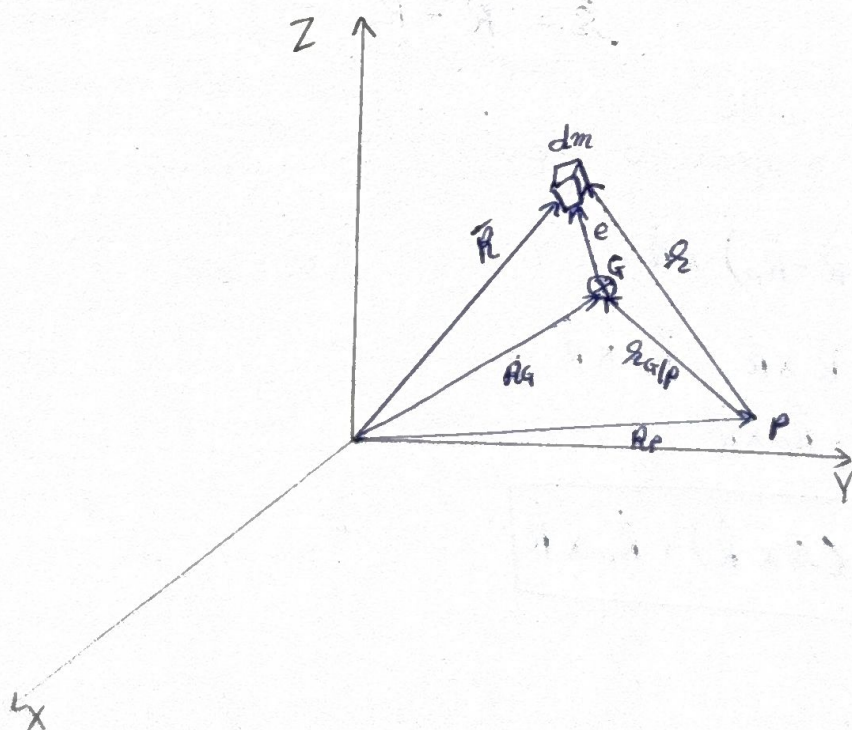
$$\boxed{\int \vec{r} \times \dot{\vec{r}} dm = \vec{H}_P}$$

$\vec{H}_P$ is the angular momentum of the body relative to point P.

$$M_{P_{net}} = \frac{d}{dt} \vec{H}_P + \dot{\vec{R}}_P \times \int \vec{r} dm$$

$\rightarrow$ Reaction is conservation of momentum wheel

date:- 22/04/2026



not dependent on dm .

$$\boxed{M_{P_{net}} = \frac{d}{dt} \int (\vec{r} \times \dot{\vec{r}}) dm + \dot{\vec{R}}_P + \int \vec{r} dm}$$

$\rightarrow$ moment about any arbitrary point P (need to be at C.G.)

$\rightarrow$ body need not be rigid

so consider $\int_m (\vec{r} \times \dot{\vec{r}}) dm \Rightarrow \vec{H}_P$

$$H_P = \int_m \bar{r} \times \dot{R} dm$$

moment of momentum of elemental mass dm

$$H_P = \int_m \bar{r} \times \dot{R} dm$$

integral of moment of momentum of all the masses over the entire body w.r.t. point P gives $\vec{H}_P$

$$\bar{r} = \bar{r}_{G/P} + \bar{s}$$

$$\vec{H}_P = \int (\bar{r}_{G/P} + \bar{s}) \times \dot{R} dm = \underbrace{\bar{r}_{G/P} \times \int \dot{R} dm}_{H_G} + \underbrace{\int \bar{s} \times \dot{R} dm}_{H_G}$$

$$\bar{r}_{G/P} \times \frac{d}{dt} \int R dm$$

$$= \bar{r}_{G/P} \times \frac{d}{dt} (R_G m)$$

$$= \bar{r}_{G/P} \times \dot{R}_G m$$

$$\vec{H}_P = \bar{r}_{G/P} \times \dot{R}_G m + \vec{H}_G$$

$$\vec{H}_P = \vec{H}_G + \bar{r}_{G/P} \times \vec{V}_G m$$

$$\Rightarrow \vec{R} = \vec{R}_G + \bar{s} \Rightarrow \dot{\vec{R}} = \dot{\vec{R}}_G + \dot{\bar{s}}$$

$$H_G = \int \bar{s} \times \dot{R} dm$$

$$H_G = \int \bar{s} \times (\dot{\vec{R}}_G + \dot{\bar{s}}) dm$$

$$= \int (\bar{s} \times \dot{\vec{R}}_G) dm + \int (\bar{s} \times \dot{\bar{s}}) dm$$

$$H_G = \int \bar{s} \times \dot{\vec{R}}_G dm +$$

only $\bar{s}$ will depend on dm & not $\dot{\vec{R}}_G$

$$= \int \bar{s} \times \dot{\vec{R}}_G dm + \int (\bar{s} \times \dot{\bar{s}}) dm$$

$$= \underbrace{\int \bar{s} dm}_0 \times \dot{\vec{R}}_G + \int (\bar{s} \times \dot{\bar{s}}) dm$$

not dependent on elemental dm

$$\vec{H}_G = \int \bar{s} \times \dot{\bar{s}} dm$$

$$H_G = \int \bar{s} \times (\dot{\vec{R}} dm)$$

$$H_{G_{rel}} = \int \bar{s} \times (\dot{\bar{s}} dm)$$

$$H_{G_{new}} = H_{G_{rel}}$$

only at C.G.

$$H_P = \int \vec{r} \times \dot{\vec{R}} dm \rightarrow \text{substitute } \vec{R} = \vec{R}_P + \vec{r}$$

$$H_P = H_{P_{rel}} + \vec{r}_{G|P} \times m \vec{V}_P$$

$$\int \vec{r} \times \dot{\vec{r}} dm$$

by equating this & previous H_P equation

$$\vec{r}_{G|P} \times m \vec{V}_G + H_G = H_{P_{rel}} + \vec{r}_{G|P} \times m \vec{V}_P$$

$$\therefore H_G = H_{P_{rel}} + \vec{r}_{G|P} \times (\vec{V}_P - \vec{V}_G) m$$

$$H_G = H_{P_{rel}} + \vec{r}_{G|P} \times m \vec{V}_{P/G}$$

$$\therefore H_{P_{rel}} = H_G + \vec{r}_{G|P} \times \underbrace{V_{P/G}}_{-V_{P/G}} m$$

$$M_{P_{net}} = \frac{d}{dt} H_P + \vec{R}_P \times \int \dot{\vec{R}} dm$$

$$M_{P_{net}} = \dot{H}_P + \vec{R}_P \times \dot{\vec{R}}_{G \rightarrow P} m$$

$$M_{P_{net}} = \dot{H}_P + \underbrace{V_P \times V_G}_{0 \text{ in case 1}} m$$

Case 1

P at rest in inertial space

$$M_{P_{net}} = \dot{H}_P$$

Case 2

P → point placed at centre of mass 'G'

$$M_{P_{net}} = M_{G_{net}} = \dot{H}_P = \dot{H}_G$$

$\Delta H = 0 \rightarrow$ conservation of angular momentum

$$\int_{t_1}^{t_2} M_{G_{net}} dt = \int_{t_1}^{t_2} \dot{H}_G \cdot H_{G_2} - H_{G_1} \Rightarrow \text{Angular impulse} \rightarrow \Delta H$$

$$M_{Pnet} = \dot{H}_P + \bar{V}_P \times m \bar{V}_G$$

$$H_P = \bar{H}_G + \bar{r}_{G/P} \times m \bar{V}_G$$

$$M_{Pnet} = \frac{d}{dt} [\bar{H}_G + \bar{r}_{G/P} \times m \bar{V}_G] + \bar{V}_P \times m \bar{V}_G$$

$$M_{Pnet} = \dot{\bar{H}}_G + \bar{r}_{G/P} \times m \bar{a}_G \rightarrow \text{acceleration of point G.}$$

* Consider the body is rigid

$\bar{S}$ with respect to Centre of mass does not change with time

$\bar{S} \rightarrow$ does not change with time

* Recall for a rigid body rotation

$$H_G = \int \bar{S} \times \dot{\bar{S}} dm$$

$$H_G = \int \bar{S} \times (\bar{\omega} \times \bar{S}) dm$$

only for rigid body



$$\frac{d\bar{S}}{dt} = \bar{\omega} \times \bar{S}$$

(પરિભ્રમણ ગતિ)

$\bar{S}$ સમજાવી રાખીએ જરૂર છે, આપણા case માં $\bar{S}$ એ $\bar{S}$ છે.

$\rightarrow$ moment with respect to a fixed point P.

$$H_P = \int \bar{r} \times \dot{\bar{R}} dm$$

$$\dot{\bar{R}} = \bar{\omega} \times \bar{R}$$

$$\bar{R} = \bar{R}_P + \bar{r}$$

$$\dot{\bar{R}} = \dot{\bar{R}}_P + \dot{\bar{r}}$$

$$H_P = \int \bar{r} \times (\bar{\omega} \times \bar{r}) dm$$

$$\rightarrow M_{Pnet} = \dot{H}_P + \dot{\bar{R}}_P \times m \bar{R}_G$$

$$M_{Pnet} = \dot{H}_P + \bar{V}_P \times m \bar{V}_G$$

* P - fixed in space, $\bar{V}_P = 0$

P - coincides with G

$$M_{Gnet} = \dot{H}_G \rightarrow \text{for a rigid body}$$

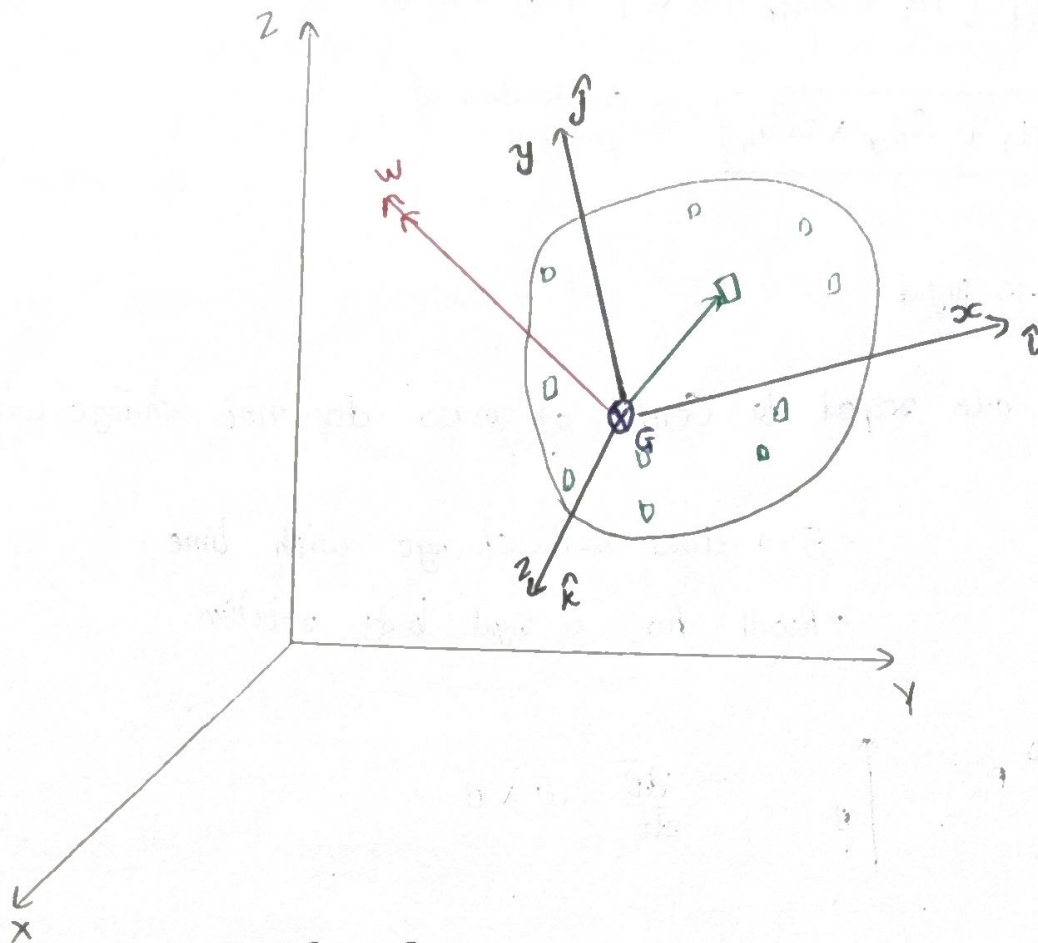
$$H_G = \int \bar{S} \times (\bar{\omega} \times \bar{S}) dm$$

$$\vec{S} \times (\vec{\omega} \times \vec{S})$$

$$\text{for } \vec{A} \times (\vec{B} \times \vec{C})$$

$$= b(a \cdot c) - c(a \cdot b)$$

$$\text{So } \vec{S} \times (\vec{\omega} \times \vec{S}) = \omega S^2 - S(\omega \cdot S)$$



$$\vec{S} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\vec{S} \times (\vec{\omega} \times \vec{S}) = (\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})(x^2 + y^2 + z^2) - (x\hat{i} + y\hat{j} + z\hat{k})(\omega_x x + \omega_y y + \omega_z z)$$

$$\therefore \int \vec{S} \times (\vec{\omega} \times \vec{S}) dm = \int [(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k})(x^2 + y^2 + z^2) - (x\hat{i} + y\hat{j} + z\hat{k})(\omega_x x + \omega_y y + \omega_z z)] dm$$

$$\begin{aligned} \therefore \int \vec{S} \times (\vec{\omega} \times \vec{S}) dm &= \int [(y^2 + z^2)\omega_x - x y \omega_y - x z \omega_z] \hat{i} dm + \int [-y x \omega_x + (x^2 + z^2)\omega_y - y z \omega_z] \hat{j} dm \\ &+ \int [-z x \omega_x - z y \omega_y + (x^2 + y^2)\omega_z] \hat{k} dm \end{aligned}$$

$$H_G = \int_m \vec{S} \times (\vec{\omega} \times \vec{S}) dm \quad H_G \text{ is a vector and we are representing it in small } xyz \text{ coordinate}$$

$$H_G = H_x\hat{i} + H_y\hat{j} + H_z\hat{k}$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_x & I_{xy} & I_{xz} \\ I_{yx} & I_y & I_{yz} \\ I_{zx} & I_{zy} & I_z \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

[I]

$$I_x = \int (y^2 + z^2) dm, \quad I_{xy} = - \int xy dm, \quad I_{xz} = - \int xz dm, \quad I_{yx} = - \int yx dm$$

$$I_y = \int (x^2 + z^2) dm, \quad I_{yz} = - \int yz dm, \quad I_{zx} = - \int zx dm, \quad I_{zy} = - \int zy dm, \quad I_z = \int (x^2 + y^2) dm$$

if we consider symmetric planes only.

$$H_x = I_x \omega_x, \quad H_y = I_y \omega_y, \quad H_z = I_z \omega_z$$