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Introduction

Boundary Layer theory ⟶ High Re Viscous flow.

I) Governing eq.
- General
- Newtonian fluid
- Specific form

II) Laminar BLT
- Incompressible BL
- Compressible BL

III) Turbulent Flow
- Introduction
- Incompressible BL

(IV) Transition

References

1. Kundu - FM
2. White - Viscous Flow
3. Sci - BLT (7ed)
4. Panton - Incompressible Flow
5. Pope - Turbulent Flow.

Evaluation

Mid - 30

End - 50

Internal - 20
- Assignment → 1 in control volume analysis.
- class test
- coding (1)
- CFD

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Newton's Laws

I law

→ Object at rest

→ Object at constant velocity

(i) Explains natural motion of bodies

(ii) Inertial frame of reference

* won't change unless otherwise external force act on it (unbalanced force).

II law

In inertial frame of reference, force is the rate of change of momentum

Galilean Invariance

* Dynamics are same in all inertial frames. $\left(V, \frac{1}{2}mv^2, mv\right)$ Dependent on FOR
* Any quantity that depends only on velocity differs with inertial frame

$a = \frac{dv}{dt} = \frac{dv}{dx}$ independent of FOR

(difference in velocity)

⇒ FOR is needed for dynamics

⇒ Coordinate system is used for our convinience.

For a given FOR, infinitely many coordinate system is possible.

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Distinction b/w solid & liquid ?

viscoelastic :- Deborah number (De) = $\frac{\text{relaxation time}}{\text{Time of observation}}$

De << 1 – Fluid

De >> 1 – solid

Macroscopic energy – Mechanics

Microscopic energy – Thermodynamics

Name	Assumption	considered	G.E	Nature of eq.
1) Point mass (or) Finite no. of point mass	Neglect shape & size	—	Balance of linear momentum (OR) Balance of total linear energy (KE+PE)	Independent variable t ODE
2) Rigid body	Deformation	Shape & size	* —do— * Balance angular momentum (OR) Balance of total energy (KE+Angular+PE)	Independent variable t ODE
3) Continuum Mechanics → Solid mechanics, Fluid mechanics	—	Deformation	* Conservation of mass * Bal. linear and angular momentum * Bal. total energy (I law) * Thermodynamic II law	Independent $\underline{x}$, t PDE

Coordinate invariance

* Natural law, geometric properties of physical objects are independent of coordinate system. Thus, tensor is introduced

Objectivity

The principle that intrinsic properties of materials should not change because of observer motion (move or rotate)

TENSORS

→ Scalar
→ Vector
→ Tensor

Representation

* Symbolic
  No need for cosine
* Indicial – cosine is needed
* Mixed → $\sum_{i=1}^{3} a_i e_i = a_i e_i = a_i' \hat{e}_i$
* Matrix.

Number of components = D^P

D → Dimension of space
P → order of tensor

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Order of tensor → No. of free indices.

Manipulate indicial notation

Eg. $a_i = Q_{im} b_m$ & $b_i = S_{im} b_m = S_{in} b_n$

$a_i = Q_{im} S_{mn} b_n$ ← $b_m = S_{mn} b_n$

→ **Isotropic tensors**

★ Kronecker delta

$$\delta_{ij} = \hat{e}_i \hat{e}_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Properties

1) $\delta_{ij} = \delta_{ji}$

2) $\delta_{ij}\delta_{kl} \neq \delta_{ik}\delta_{jl}$

3) $\delta_{ij}\delta_{jk} = \delta_{ik}$

4) $\delta_{ii} = 3$

5) $\boxed{\delta_{ij} u_j = u_i}$

★ Levi-Civita / Permutation

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{cyclic} \\ 0 & \text{any indice repeat} \\ -1 & \text{anticyclic} \end{cases}$$

$$\boxed{\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk} \hat{e}_k}$$

$$\hat{e}_i \cdot (\hat{e}_j \times \hat{e}_k) = \hat{e}_i (\varepsilon_{\underset{jk}{ijm}} \hat{e}_m) = \varepsilon_{\underset{jk}{ijm}} (\hat{e}_i \hat{e}_m) = \varepsilon_{jkm}\delta_{im} = \varepsilon_{jki} = \varepsilon_{ijk}$$

Properties

1) $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$

2) $\varepsilon_{ijk}\,\varepsilon_{ijn} = 2\delta_{kn}$

3) $\varepsilon_{ijk}\,\varepsilon_{ijk} = 6$

4) $\varepsilon_{ijk}\,\delta_{jk} = 0$

Vector operation (for mixed notation)

1) Addition / Subtraction

$\underline{a} \pm \underline{b} = \underline{c}$ → Symbolic

$a_i \pm b_i = c_i$ → Indicial

$a_i\hat{e}_i \pm b_i\hat{e}_i = c_i\hat{e}_i$ → Mixed

2) Dot product / Inner product

$\underline{a}\cdot\underline{b}$ → Symbolic

$a_j b_j$ → Indicial

$(a_i\hat{e}_i)\cdot(b_j\hat{e}_j)$ ✓

3) Vector product

$\underline{a}\times\underline{b}$

$(a_i\hat{e}_i)\times(b_j\hat{e}_j) = a_i b_j(\hat{e}_i\times\hat{e}_j)$

$= a_i b_j\,\varepsilon_{ijk}\hat{e}_k$

mth component

$(\underline{a}\times\underline{b})_m = \hat{e}_m\cdot(\underline{a}\times\underline{b})$

$= \hat{e}_m\cdot(\varepsilon_{ijk}\,a_i b_j\hat{e}_k)$

$= \varepsilon_{ijk}a_i b_j\,\delta_{mk}$

$= \varepsilon_{ijm}\,a_i b_j$

1) $(\underline{a}\times\underline{b})\times\underline{c} = (\underline{a}\cdot\underline{c})\cdot\underline{b} - (\underline{b}\cdot\underline{c})\cdot\underline{a}$

2) $(\underline{u}\times\underline{v})\cdot(\underline{u}\times\underline{v})$

4) Triple product

$\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\cdot\underline{b} - (\underline{a}\cdot\underline{b})\cdot\underline{c}$

$= (a_i\hat{e}_i)\times(\varepsilon_{jkm}\,b_j c_k\hat{e}_m)$

$= \varepsilon_{jkm}\,a_i b_j c_k\,\varepsilon_{imn}\hat{e}_n$

$= \cancel{\varepsilon_{jkm}\,a_i b_j}$

$= (\delta_{jn}\delta_{ki} - \delta_{ji}\delta_{kn})\,a_i b_j c_k\hat{e}_n$

$= a_k b_n c_k\hat{e}_n - a_i b_i c_n\hat{e}_n$

$= (\underline{a}\cdot\underline{c})\cdot\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$

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Tensor of order 2

$\underline{\underline{A}}$ - Symbolic

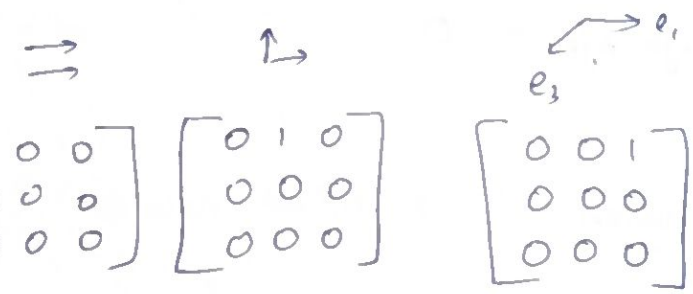
A_{ij} - indicial

$A_{ij}\,\hat{e}_i \otimes \hat{e}_j$ - Symbolic/Mixed

$[A]$ - Matrix

Basis

$e_1 \otimes e_1$ $\quad$ $e_1 \otimes e_2$ $\quad$ $e_1 \otimes e_3$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_{ij}\, e_i \otimes e_j = A_{11}\, e_1 \otimes e_1 + A_{12}\, e_1 \otimes e_2 \ldots + A_{23}\, e_2 \otimes e_3 + A_{33}\, e_3 \otimes e_3$$

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Operation b/w unit tensor

Dot product

1) $(e_i \otimes e_j) \cdot e_k = \hat{e}_i (e_j \cdot e_k) = \hat{e}_i\, \delta_{jk}$

2) $e_i \cdot (e_j \otimes e_k) = (e_i \cdot e_j)\, \hat{e}_k = \delta_{ij}\, \hat{e}_k$

3) $(e_i \otimes e_j) \cdot (e_m \otimes e_n) = (e_j \cdot e_m)(e_i \otimes e_n) = \delta_{jm} (e_i \otimes e_n)$.

Double dot

~~Cross~~ product / scalar product

$$(e_i \otimes e_j) : (e_m \otimes e_n) = (e_j \cdot e_m)(e_i \cdot e_n) = \delta_{jm}\, \delta_{in}$$

Cross product

$$(e_i \otimes e_j) \times e_k = e_i \otimes (e_j \times e_k)$$

$$= e_i \otimes \varepsilon_{jkm}\, e_m$$

$$= \varepsilon_{ijn}\, \varepsilon_{jkm}\, e_i \otimes e_m .$$

Tensor product

$e_i \otimes e_j$

Operation	Resulting order
$\otimes$	Σ
$\times$	$\Sigma - 1$
$\cdot$	$\Sigma - 2$
$:$	$\Sigma - 4$

Dot product b/w two tensors

$\underline{\underline{A}} \cdot \underline{\underline{B}} = [A_{ij}\, e_i \otimes e_j] \cdot [B_{mn}\, e_m \otimes e_n]$

~~$= A_{ij}\, e_i \otimes$~~

$= A_{ij}\, B_{mn}\, (e_i \otimes e_j) \cdot (e_m \otimes e_n)$

$= A_{ij}\, B_{mn}\, \delta_{ij}\, e_i \otimes e_n$

$= A_{ij}\, B_{jmn}\, e_i \otimes e_n$

$(\underline{\underline{A}} \cdot \underline{\underline{B}})_{ij} = A_{ik}\, B_{kj}$

$\underline{\underline{A}} \cdot \underline{b} = A_{ij}(e_i \otimes e_j) \cdot b_m\, e_m$

$= A_{ij}\, b_m\, (e_j \cdot e_m)\, e_i$

$= A_{ij}\, b_m\, \delta_{jm}\, e_i$

$= A_{ij} b_j\, e_i$

$(\underline{\underline{A}} \cdot \underline{b})_i = A_{ij}\, b_j$

$\underline{b} \cdot \underline{\underline{A}} =$ (HW)

$(\underline{b} \cdot \underline{\underline{A}})_i = b_j A_{ji}$

$= A^T_{ij}\, b_j$

$(\underline{a} \otimes \underline{b})\, \underline{c} = (\underline{b} \cdot \underline{c})\, \underline{a}$

$\underline{a}\, (\underline{b} \otimes \underline{c}) = (\underline{a} \cdot \underline{b})\, \underline{c}$

$\underline{\underline{T}} : (\underline{a} \otimes \underline{b}) = (\underline{\underline{T}} \cdot \underline{a}) \cdot \underline{b}$

Symbolic	Tensor Component	Matrix
$\underline{c} = \underline{a} \cdot \underline{b}$	$c = a_i b_i$	$[c] = \{a\}^T \{b\}$
$\underline{\underline{A}} = \underline{a} \otimes \underline{b}$	$A_{ij} = a_i b_j$	$[A] = \{a\}\{b\}^T$
$\underline{b} = \underline{\underline{A}}\, \underline{x}$	$b_i = A_{ij} x_j$	$\{b\} = [A]\{x\}$
$\underline{b} = \underline{x}\, \underline{\underline{A}}$	$b_j = x_i A_{ij}$	$\{b\} = \{x\}^T [A]$
$\underline{\underline{D}} = \underline{\underline{A}}\, \underline{\underline{B}} \cdot \underline{\underline{C}}$	$D_{ij} = A_{im} B_{mn} C_{nj}$	$[D] = [A][B][C]$
$\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}$	$C_{ij} = A_{im} B_{mj}$	$[C] = [A][B]$
$\underline{\underline{C}} = \underline{\underline{A}} \cdot \underline{\underline{B}}^T$	$C_{ij} = A_{im} B_{jm}$	$[C] = [A][B]^T$
$\underline{c} = \underline{a} \cdot \underline{\underline{A}} \cdot \underline{b}$	$c = a_i A_{ij} b_j$	$c = \{a\}^T [A] \{b\}$

Special types of tensor

$$A_{ij} = \underbrace{\frac{1}{2}\left[A_{ij} + A_{ji}\right]}_{\text{Sym.}} + \underbrace{\frac{1}{2}\left[A_{ij} - A_{ji}\right]}_{\text{Skew-sym.}}$$

Skew symmetric / Anti-Symmetric

$$R_{ij} = -R_{ji} = \begin{bmatrix} 0 & -b_3 & +b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

* Since there are three independent components, it can also be represented by ~~the~~ a pseudo-vector / axial vector. → Not independent of coordinate system (Ω_k).

$$\Omega_k = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$R_{ij} = -\varepsilon_{ijk}\,\Omega_k \quad \text{(HW)}$$

$$\Omega_k = \frac{1}{2}\,\varepsilon_{ijk}\,R_{ij}$$

Why go for pseudo-vector?

$$\underline{\underline{R}}\,\underline{x} = R_{ij}\,(\hat{e}_i \otimes \hat{e}_j)\,(x_m \hat{e}_m) = R_{ij}\,x_m\,\delta_{jm}\,\hat{e}_i = R_{ij}x_j\,\hat{e}_i$$

$$= -\varepsilon_{ijk}\,\Omega_k\,x_j\,\hat{e}_i$$

$$= -\varepsilon_{ijk}\,x_j\,\Omega_k\,\hat{e}_i$$

$$= -(\underline{x} \times \underline{\Omega})$$

$$\boxed{\underline{\underline{R}}\,\underline{x} = \underline{\Omega} \times \underline{x}}$$

Pseudovector is helpful in computational efficiency

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Rotation of basis vectors → we are still representing the same vector → ON rule

Rotation about $\hat{e}_3$.

$$\hat{e}_1 = \cos\theta\,\hat{e}_1' - \sin\theta\,\hat{e}_2'$$

$$\hat{e}_2 = \sin\theta\,\hat{e}_1' + \cos\theta\,\hat{e}_2'$$

$$\begin{Bmatrix}\hat{e}_1\\ \hat{e}_2\end{Bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}\hat{e}_1'\\ \hat{e}_2'\end{Bmatrix}$$

$$\{\hat{e}_i\} = [T]\{\hat{e}_i'\}$$

$$\{\hat{e}_i'\} = [T]^T\{\hat{e}_i\}$$

Thus,

$$\hat{e}_i' = \underline{\underline{T}}\,\hat{e}_i = T_{ji}\,\hat{e}_j$$

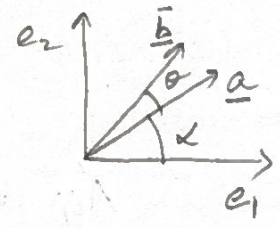
But, $T_{ij} = (e_i \cdot \underline{\underline{T}} e_j) = \hat{e}_i \cdot \hat{e}_j'$.

$$T = \begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}$$ (ON)

Rotation of a vector → the vector itself changes.

$$\underline{a} = |a|\cos\alpha\,\hat{e}_1 + |a|\sin\alpha\,\hat{e}_2$$

$$= a_1\hat{e}_1 + a_2\hat{e}_2$$



$$\underline{b} = |b|\cos(\theta+\alpha)\hat{e}_1 + |b|\sin(\theta+\alpha)\hat{e}_2$$

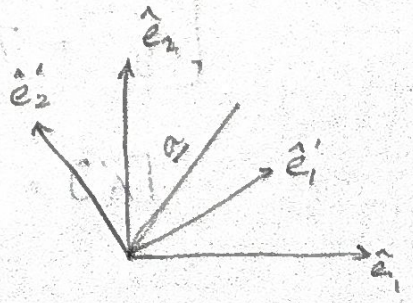
$$\begin{Bmatrix}b_1\\ b_2\end{Bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\end{Bmatrix}$$

$$\{b_i\} = [T]\{a_i\}$$

$$\underline{b} = \underline{\underline{T}}\,\underline{a}$$

Thus, $\boxed{b_i = T_{ij}\,a_j}$

Representation in Indicial form (Component of a vector in a rotated basis)



Consider the vector $\underline{a}$.

$$a_i' = \hat{e}_i' \cdot \underline{a}$$

$$a_i = \hat{e}_i \cdot \underline{a}$$

Now, $a_i' = \hat{e}_i' \cdot \underline{a} = \underline{a} \cdot \hat{e}_i'$

$$= (a_j\hat{e}_j).(\underline{\underline{T}}\hat{e}_i) = a_j(e_j \cdot \underline{\underline{T}} e_i) = a_j\,T_{ji}$$

(or)

$$a'_i = \underline{a} \cdot \hat{e}'_i = (a_j \hat{e}_j) \cdot (\underline{\underline{T}} \hat{e}_i)$$

$$= a_j \hat{e}_j \cdot (T_{mi} \hat{e}_m)$$

$$= a_j T_{mi} \delta_{jm}$$

$$= a_j T_{ji}$$

Thus, $\boxed{a'_i = T_{ji} a_j}$ $\quad \{a'\} = [T]^T \{a\}$

Consider

$$\underline{b} = \underline{\underline{T}}\, \underline{a}$$

Now $b_i = \hat{e}_i \cdot \underline{b} = \hat{e}_i \cdot \underline{\underline{T}}\, \underline{a} = \hat{e}_i \cdot (\underline{\underline{T}} a_j \hat{e}_j) = a_j (\hat{e}_i \underline{\underline{T}} \hat{e}_j) = a_j T_{ij}$

$$\boxed{b_i = T_{ij} a_j}$$

Tensor components

$A_{ij} = e_i (\underline{\underline{A}} e_j)$ $\qquad$ $A'_{ij} = \hat{e}'_i \cdot (\underline{\underline{A}} \hat{e}'_j)$

Now, $A'_{ij} = (T\hat{e}_i) \cdot A (T e_j)$

$= (T_{mi} \hat{e}_m) \cdot A \cdot (T_{nj} \hat{e}_n)$

$= T_{mi} T_{nj} (\hat{e}_m \cdot A \hat{e}_n)$

$= T_{mi} T_{nj} A_{mn}$

$$\boxed{A'_{ij} = T_{mi} A_{mn} T_{nj}}$$ (ON)

$$[A'] = [T]^T [A] [T]$$

$A'_{ij} = T_{mi} T_{nj} A_{mn}$ Tensor order 2

$S'_{ijk} = T_{mi} T_{nj} T_{ok} S_{mno}$ Tensor order 3

$a'_i = T_{ji} a_j$ Tensor order 1

$\alpha' = \alpha$ Tensor 0. (Scalar)

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TENSOR CALCULUS

Field – any quantity that depends on space. $u(\underline{x}, t)$

Derivatives

Time derivative : $\frac{d}{dt}(a_i \hat{e}_i) = \frac{da_i}{dt}\hat{e}_i$

$$\frac{d}{dt}[A_{ij}\, e_i \otimes e_j] = \frac{dA_{ij}}{dt}\hat{e}_i \otimes \hat{e}_j$$

Space derivative :

★ Gradient – $\underline{\nabla} = e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}$

Max. Rate of change in basis direction

$$\underline{\nabla} = e_i \frac{\partial}{\partial x_i}$$

Gradient of scalar

$$\underline{\nabla}\phi = e_i \frac{\partial \phi}{\partial x_i}$$

Directional derivative ($\hat{e}_n$)

$$\nabla_{\hat{e}_n}\phi = \hat{e}_n \cdot \nabla\phi = \frac{\partial \phi}{\partial n}$$

$$d\phi = \phi(\underline{x} + d\underline{x}) - \phi(\underline{x})$$

$$= d\underline{x} \cdot \nabla\phi = |dx|\, \hat{e}_x \cdot \nabla\phi = |dx|\frac{d\phi}{dx}.$$

↑ Taylor series expansion

Gradient of a vector (Vector → Tensor)

$\frac{\partial a_i}{\partial x_j}$

$$\underline{\nabla} \otimes \underline{a} = e_j \frac{\partial}{\partial x_j} a_i$$

$$(\underline{\nabla} \otimes \underline{a})_{ij} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}$$

(Row, column)

Any tensor can be written in sum of symmetric and skew symmetric.

∇a = sym + skewsym

sym: deviatoric (shear), volumetric (div)

skewsym: (curl)

★ Divergence (Tensor → vector)

$\frac{\partial a_i}{\partial x_i}$

$$\underline{\nabla} \cdot \underline{a} = e_j \frac{\partial}{\partial x_j}(a_i e_i) = \frac{\partial a_i}{\partial x_j}\delta_{ij} = \frac{\partial a_i}{\partial x_i}$$

$$\underline{\nabla} \cdot \underline{\underline{A}} = e_j \frac{\partial}{\partial x_j}(A_{mn}\, e_m \otimes e_n) = \frac{\partial}{\partial x_j} A_{mn}\, \delta_{jm} e_n = \frac{\partial A_{jn}}{\partial x_j} e_n$$

$$(\underline{\nabla} \cdot \underline{\underline{A}})_{ij} = \frac{\partial a_{ij}}{\partial x_i}$$

★ Curl

$$\underline{\nabla} \times \underline{u} = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \hat{e}_i$$

$$(\underline{\nabla} \times \underline{u})_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

★ Laplace

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_j \partial x_j} \qquad \nabla^2 \underline{u} = \frac{\partial^2 u_i}{\partial x_j \partial x_j} \hat{e}_i$$

Integral theorem

Fundamental theorem of calculus

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a)$$

Integral of the derivative = value at the boundary.

Integral of volume (derivative of) = value at surface

" surface = " curve

" curve = " end points

General

$$\int_{\forall} \frac{\partial T_{jk\ldots}}{\partial x_i} d\forall = \oint_A n_i T_{jk\ldots} dA$$

(Generalized Kelvin - Stokes theorem)

⇒ Gradient

$$\int_{\forall} \frac{\partial \phi}{\partial x_i} d\forall = \oint_A n_i \phi dA \qquad \int_{\forall} \nabla \phi \, d\forall = \int_A \hat{n} \phi dA$$

$$\int_{\forall} \frac{\partial u_j}{\partial x_i} d\forall = \oint_A n_i u_j dA \qquad \int_{\forall} \underline{\nabla} \cdot \underline{u} \, d\forall = \int_A \hat{n} \underline{u} dA$$

$$\int_{\forall} \frac{\partial T_{jk}}{\partial x_i} d\forall = \int n_i T_{jk} dA \qquad \int_{\forall} \underline{\nabla} \underline{\underline{T}} d\forall = \int_A \hat{n} \underline{\underline{T}} dA$$

⇒ Divergence

$$\int_V \frac{\partial u_i}{\partial x_i} dV = \int_A n_i u_i dA$$

$$\int_V \frac{\partial T_{ij}}{\partial x_i} dV = \int_A n_i T_{ij} dA$$

$$\int_V \nabla \cdot \underline{u} \, dV = \int_A (\hat{n} \cdot \underline{u}) \, dA$$

$$\int_V \nabla \cdot \underline{\underline{T}} \, dV = \int_A (\hat{n} \cdot \underline{\underline{T}}) \, dA$$

⇒ curl

$$\int_V \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} dV = \int_A n_j \varepsilon_{ijk} u_k \, dV$$

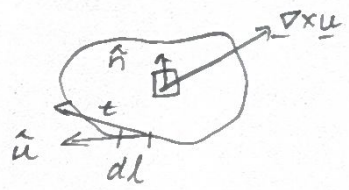
$$\int_V (\nabla \times \underline{u}) \, dV = \int_A (\hat{n} \times \underline{u}) \, dA$$

Stokes theorem

$$\iint_A \hat{n} \cdot (\nabla \times \underline{u}) \, dA = \oint_C \underline{t} \cdot \underline{u} \, dl$$

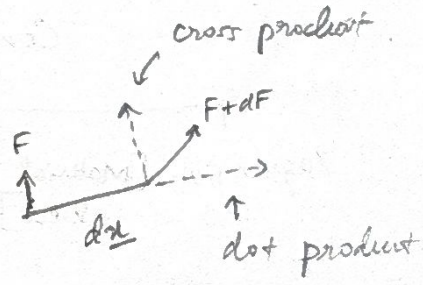
(tangent vector)

$$\iint_A \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} n_i \, dA = \oint_C t_i u_i \, dl$$



Divergence

↳ average rate of change along the directions
pure volumetric change.



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HW

Q) what are the invariants of tensor order 2.

Q) Eigen value & Eigen vector,

Q) ↳ Advantage / why it is used. (Decoupling)

MODELLING OF FLUID FLOW

Fluid Flow

Particle	Continuum
- Apply laws in microscopic level - Average the result to get macroscopic quantity $\underline{u}, P, T, \rho$	- Average - Apply laws in macroscopic level ⇓ - Result in macroscopic variable.

Valid $Kn = \frac{\lambda}{L} < 0.01$ (Knudsen number; λ: molecular mean free path; L: characteristic length)

λ @STP (air) $\sim 10^{-9}$ m

Continuum

Lagrangian [Material view]	Euler [Field view]
Fixed mass / closed system / Material volume (Nothing moves in/out of it)	Open system / control volume
* Infinitesimal material volume (Differential) (fluid particle)	* Infinitesimal control volume
* Finite material volume (Integral)	* Finite control volume

Material derivative

RTT

- All natural laws (mechanics & thermodynamics) deals with fixed mass only

Types of derivative

1) Fixed location $b(t)$

$$\frac{db}{dt} = \frac{db}{dt}$$

2) particle moves with some velocity $\vec{u}$ (vector) $b(\vec{x}, t)$

$$\frac{db}{dt} = \frac{\partial b}{\partial t} + \frac{\partial b}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial b}{\partial x_2}\frac{dx_2}{dt} + \cdots$$

$$\frac{db}{dt} = \frac{\partial b}{\partial t} + u_j^a \frac{\partial b}{\partial x_j} = \frac{\partial b}{\partial t} + (\underline{u}^a \cdot \nabla) b$$

3) particle moves with fluid velocity / drifting with flow.

$b(\underline{x}, t)$ $\quad \underline{u} \rightarrow$ fluid velocity

$$\frac{db}{dt} = \frac{\partial b}{\partial t} + (\underline{u} \cdot \nabla) b = \frac{\partial b}{\partial t} + u_j \frac{\partial b}{\partial x_j} = \frac{Db}{Dt}$$

(Material / Substantial Derivative)

Acceleration,

$$\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}$$

Euler

local $\quad$ advective

$$= \frac{\partial u}{\partial t} + \nabla \cdot \frac{|u|^2}{2} - (\underline{u} \times \nabla \times \underline{u})$$

$$= \frac{\partial u}{\partial t} + \nabla\left(\frac{|u|^2}{2}\right) - (\underline{u} \times \underline{w})$$

Convection = Conduction + Advection

$\nabla^2 \underline{u}$ – linear

$(\underline{u} \cdot \nabla)\underline{u}$ – non-linear

→ Euler equation is non-linear,

→ In incompressible aerodynamics, it was assumed the flow to be irrotational.

Thus, linear.

$\phi \Rightarrow u = \frac{\partial \phi}{\partial x}, \; v = \frac{\partial \phi}{\partial y}$ Kinematics.

$\nabla \cdot \underline{u} = 0 \Rightarrow \nabla^2 \phi = 0.$

→ Kinematics and dynamics are decoupled.

↓ still non-linear.

$P_0 = P + \frac{1}{2}(u^2)$

Linear	Non-linear
* Superposition is applicable	* No superposition applicable
* Unique Infinite No solution	* -do- Finite solution.
* Standard method	* No standard method. * Sensitive to initial condition.

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Reynolds Transport Theorem

B → Extensive property

b → intensive property = B/m

I

$$\frac{DB}{Dt} = \frac{D}{Dt}\int_{MV}(\rho b\, dV) = \frac{d}{dt}\int_{CV}\rho b\, dV + \iint_{CS}\rho\, b\,(\underline{u}-\underline{u}_s)\hat{n}\, dA$$

$\underline{u}$ → Flow velocity

$\underline{u}_s$ → control surface velocity.

II

$$\frac{DB}{Dt} = \frac{D}{Dt}\int_{MV}\rho b\, dV = \int_{CV}\frac{\partial(\rho b)}{\partial t}\, dV + \underbrace{\iint_{CS}\rho b\,(\underline{u}\cdot\hat{n})\, dA}_{\text{Net flow of property through surface}}$$

Leibnitz Theorem

$$\frac{d}{dt}\int_{V(t)}\rho b\, dV = \int_{V}\frac{\partial(\rho b)}{\partial t}\, dV \overset{\ominus}{+} \int_{CS}\rho b\,\underline{u}_s\hat{n}\, dA$$

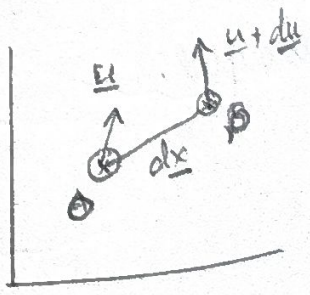
Unsteadiness in integrand — Unsteadiness in boundary limits

Kinematics

- Study of motion without its cause

Motions of the fluid element

1) Translation
2) Rotation
3) Deformation
   - Volumetric (Linear)
   - Deviatoric (Shear)



Relative motion between two points

@ O : $\underline{u}$ $\quad OP = d\underline{x}$

@ P : $\underline{u} + d\underline{u}$

Now, $\underline{u} + d\underline{u} = \underline{u} + (d\underline{x} \cdot \nabla)\underline{u}$ Taylor series truncated to 1st order

$$(\nabla \cdot \underline{u})_i = \frac{\partial u_i}{\partial x_j} \xrightarrow{\text{decompose}} \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

(Transpose)

$$\frac{\partial u_i}{\partial x_j} = \overset{\text{Sym.}}{S_{ij}} + \overset{\text{Skew-Sym.}}{R_{ij}}$$

S_{ij}: Rate of deformation tensor; R_{ij}: Rotation tensor

Again, $S_{ij} = d_{ij} + \frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$

d_{ij}: Deviatoric (shear); $\frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$: Volumetric (dilatation)

Rate of shear tensor = Change in shape, not size / volume

Volumetric tensor = Change in volume or size, not in shape

$$\Rightarrow d_{ij} = S_{ij} - \frac{1}{3}\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij}$$

$$= \begin{bmatrix} S_1 - \frac{1}{3}(S_{11} + S_{22} + S_{33}) & & \\ & S_{22} - (\ldots) & \\ & & S_3 - (\ldots) \end{bmatrix}$$

$$S_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$$

$$S_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) \cdots$$

$\text{tran}(d_{ij}) = 0$ //

We know that,

$$R_{ij} = -\varepsilon_{ijk}\,\Omega_k$$

From initial velocity formulation, $d\underline{u} = (d\underline{x} \cdot \nabla)\underline{u}$

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j$$

Thus,

$$du_i = [S_{ij} + R_{ij}]\,dx_j$$

$$= \left[d_{ij} + \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} + R_{ij}\right]dx_j$$

$$= \left[d_{ij} + \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]dx_j + R_{ij}\,dx_j$$

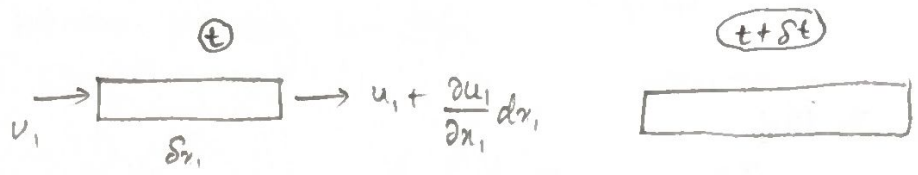
$$= \left[d_{ij} + \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right]dx_j + \frac{1}{2}\underline{\omega} \times d\underline{x}$$

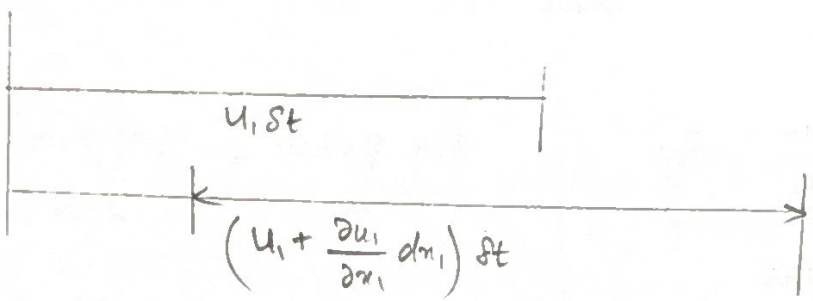
$$R_{ij}\,dx_j = \underline{\Omega} \times d\underline{x}$$

$$= \frac{1}{2}\underline{\omega} \times d\underline{x}$$

Linear deformation

$$\text{Strain (rate)} = \frac{1}{dt}\left(\frac{\Delta l}{l}\right) \text{ or } \frac{1}{dA}\left(\frac{\Delta A}{A}\right) \text{ or } \frac{1}{dV}\left(\frac{\Delta V}{V}\right)$$



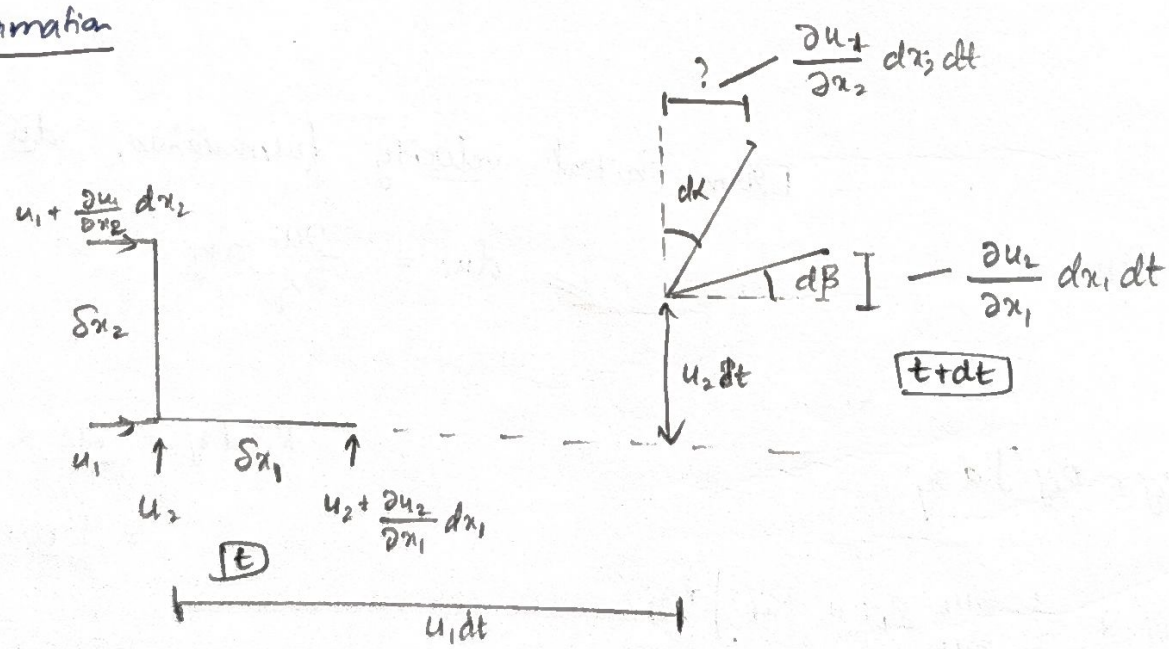


$$\text{Linear Strain rate} = \frac{1}{dt}\frac{\left(\frac{\partial u_1}{\partial x_1} dx_1 \delta t\right)}{\delta x_1} = \frac{\Delta l}{l} = \frac{\partial u_1}{\partial x_1} \quad \text{(Small deformation)}$$

Similarly, volumetric rate

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}.$$

Shear deformation



Shear strain rate: average rate of change of angle formed by mutually ⊥ lines in the fluid element.

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$S = \eta\theta$

$$du_i = \left(\frac{\partial u_i}{\partial x_j}\right) dx_j$$

↑ Velocity gradient tensor

- Deformation
  - volumetric
  - Deviatoric
- Rotation

Shear strain rate.

$$S_{12} = \frac{d}{dt}\left[\frac{(\alpha_2 - \alpha_1)}{2}\right] = \frac{d}{dt}\left[\frac{d\beta + (-d\alpha)}{2}\right] \quad \text{Average}$$

$$= \frac{d}{dt}\left[\frac{d\beta + d\alpha}{2}\right]$$

$$= \frac{1}{2}\frac{d}{dt}\left[\frac{\frac{\partial u_2}{\partial x_1} dx_1 dt}{dx_1} + \frac{\frac{\partial u_1}{\partial x_2} dx_2 dt}{dx_2}\right]$$

$$\boxed{S_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = S_{21}}$$

Rotation rate:

Average rotation rate of two mutually perpendicular lines in the fluid element.

$$R_{12} = \frac{d}{dt}\left(\frac{\alpha_2 + \alpha_1}{2}\right) = \frac{d}{dt}\left(\frac{d\beta + (-d\alpha)}{2}\right) = \frac{dt}{dt}$$

$$\boxed{R_{12} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)}$$

Vorticity, $\underline{\omega} = \underline{\nabla} \times \underline{u}$

Rotation, $\underline{\Omega} = \frac{1}{2}\underline{\omega}$

Reason for classification

→ we would like to remove rotation for stress purpose

→ Material behaviour varies based on volumetric & or deviatoric. Thus, split it also.

$$(\underline{u} + d\underline{u})_i = \underbrace{u_i}_{\text{Translation}} + \underbrace{\left(d_{ij} + \frac{1}{3}\frac{\partial u_k}{\partial x_k}\right)}_{\text{Deviatoric}\ \ \text{Volumetric}} dx_j + \underbrace{\frac{1}{2}(\underline{\omega} \times d\underline{x})_i}_{\text{Rotation}} \quad \text{(velocity at neighbouring point)}$$

Frame of Reference (FOR$_{II}$ - Rigid body motion) (Non-inertial frame)

$$\underline{u}_{II} = \underline{u}^{\wedge} + (\underline{r} \times \underline{\omega})$$

$$\underline{u}_{II} = (\underline{u}^{\wedge}, \underline{\Omega}^{\wedge}) \qquad \underline{\Omega} \rightarrow \text{angular velocity.}$$

1) Frame indifference (or) objectivity

* Quantities which are independent of FOR.

Ex. Strain rate, Stress

Change :- Vorticity & Kinetic energy

2) Galilean invariance (another inertial frame)

$$\underline{u}_{\overline{IV}} = (\underline{u}^{R})$$

* Quantities which are independent of inertial FOR

Ex. Any quantity which has Δv

Strain, Vorticity,

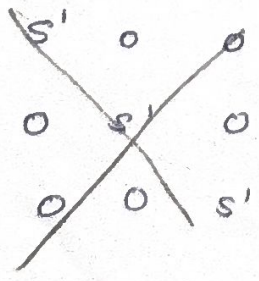
Change :- Kinetic Energy

3) Coordinate invariance

Ex. Physical or material quantities.

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} S_{11} & & \\ & S_{22} & \\ & & S_{33} \end{bmatrix}}_{\text{Volume + shear deviatoric}} + \underbrace{\begin{bmatrix} 0 & S_{12} & S_{13} \\ S_{21} & 0 & S_{23} \\ S_{31} & S_{32} & 0 \end{bmatrix}}_{\text{Pure shear}}$$

deviatoric $\begin{cases} \text{shear} \\ \text{axial force} \end{cases}$



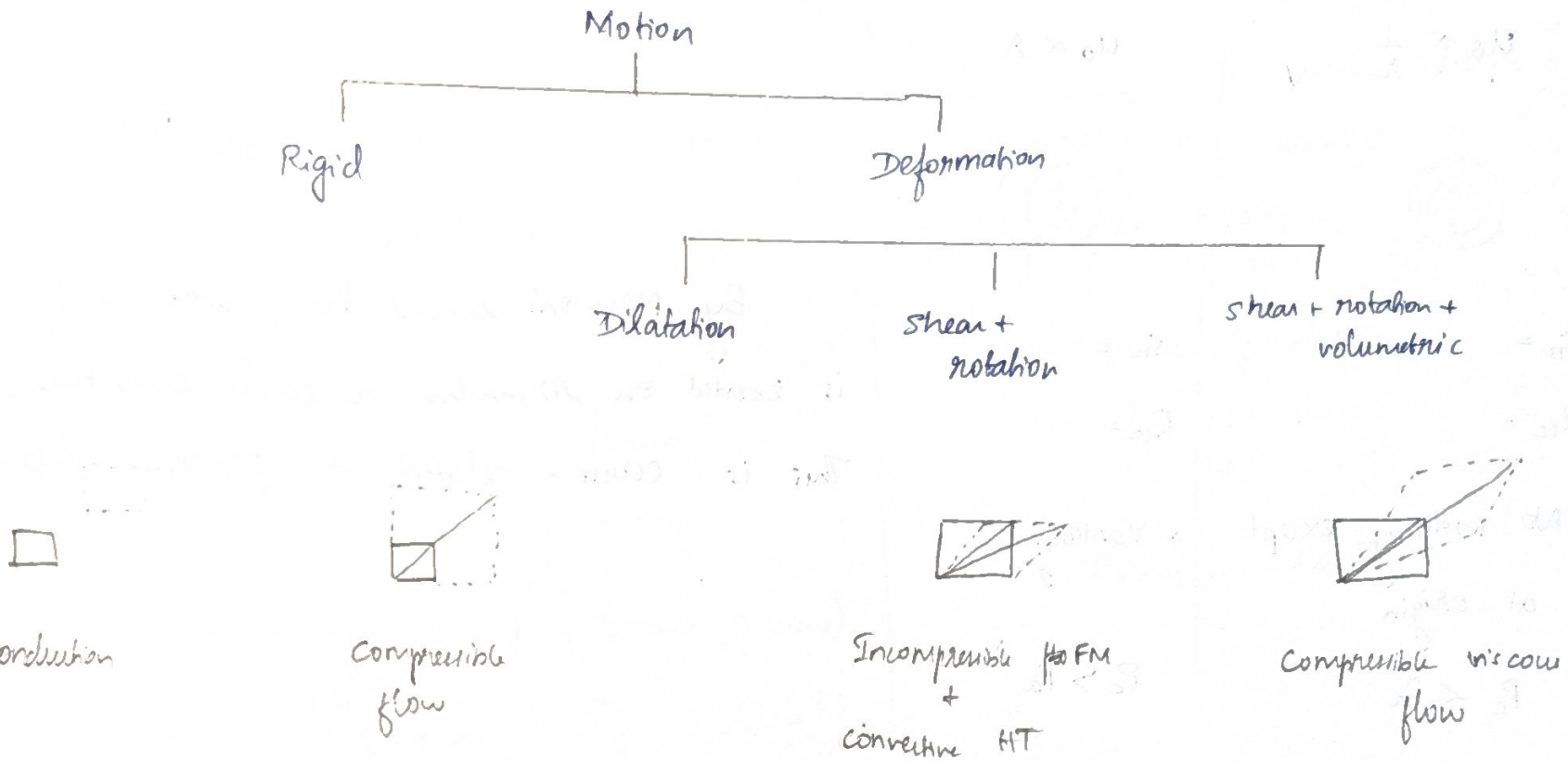
$S_{11} + S_{22} + S_{33} = 0$ Pure volumetric

$S_{11} + S_{22} + S_{33} = 0$ Pure shear deviatoric

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonal elements only — Pure volume — Pure shear deviatoric

classification of thermal stress motion based on kinematics



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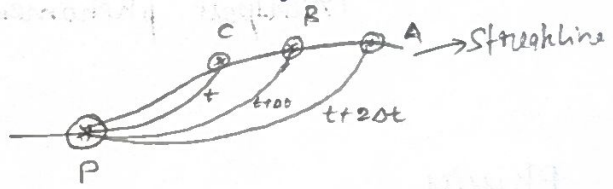
Streamline : instantaneous

Path line : time interval

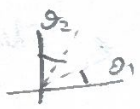
Streakline : instantaneous trace of all the fluid element that pass through a fixed point in space for a specified time interval.



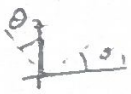
Vorticity : $\underline{\omega} = \underline{\nabla} \times \underline{u}$

circulation : $\Gamma = \oint \underline{t} \cdot \underline{u} \, dl$

$S_{12} = \frac{\theta_2 - \theta_1}{2}$



$R_{12} = \frac{\theta_2 + \theta_1}{2}$



Stream tube :

Vorticity tube : → Vorticity is the property of flow field @ each point.

→ ~~Vortex~~ Rotation of the fluid about its centre

Vortex → A loop of fluid movement.

→ Rotation about some other point.

→ Tube surface made of vorticity line

Free vortex

$u_\theta \propto \frac{1}{r}$

$S_{12} =$

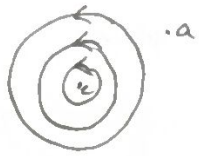
$R_{12} =$

* No vorticity except at origin

$P_c < P_a$

Forced vortex

$u_\theta \propto r$



$S_{12} =$

$R_{12} =$

* Vorticity

$P_c > P_a$ ✗

Streamline curvature argument

$P_a > P_c$

By Newton's second law, when a force is exerted on streamline to cause curvature. This is cause-related to pressure differ.

Flow & Flux

Flux — Physics / Vector calculus
Flux — Transport phenomena

$$\text{Flux} = \frac{(\ )}{A \cdot t}$$

$$\text{Flow} = \frac{(\ )}{t}$$

Vector / Tensor

Transport

Physics

* Flux is a *scalar* quantity which is integral of the normal component of area.

$$\phi_F = \int_A F \cdot \hat{n}\, dA$$

Eg.

$$\text{Mass flux} = \frac{m}{At} = \rho \underline{u}$$

$$\text{Mass flow rate} = \int_A (\rho \underline{u} \cdot \hat{n})\, dA$$

$$\text{Momentum flux} = \frac{mu}{At} = \rho \underline{u}\,\underline{u}$$

$$\text{Momentum flow rate} = \int_A \rho \underline{u}\,(\underline{u} \cdot \hat{n})\, dA$$

DYNAMICS

Nature laws

Independent of material	Dependent on material
General principle / law	Constitutive relations
→ Conservation of mass	→ Equation of State
→ Balance of linear momentum	├ TEOS Thermal
→ " " angular "	└ CEOS caloric
→ Equilibrium principle (Thermo 0 law)	→ Transport equation
→ Energy conservation (" I law)	
→ Entropy principle (" II law)	

1) Kinematic variable :- u, K.E, ω

2) Thermodynamic variable :- P, ρ, T, h, e, C_p

3) Transport variable :- μ, K, D

4) others

Eqns

Conservation ~~law~~ of mass

RTT $\quad B = m$

$b = 1$

$$\left.\frac{dm}{dt}\right|_{sys} = \frac{d}{dt}\int_{MV} \rho \, dv = \int_{CV} \frac{\partial(\rho)}{\partial t} dv + \int_{CS} \rho(\underline{u}\cdot\hat{n}) dA = 0$$

$$\int_{v} \frac{\partial \rho}{\partial t} dv + \int_{cs} \rho u_j \hat{n}_j \, dA = 0$$

$$\int_{v} \frac{\partial \rho}{\partial t} dv + \int_{cv} \frac{\partial}{\partial x_j}(\rho u_j) dA = 0 \quad \Rightarrow \int_{v} \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j}\right] dv = 0$$

According to localisation theorem,

$$\int_{\forall} L \, d\forall = 0 \quad \Rightarrow \boxed{L = 0}$$

Thus,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$$

Rewriting

I. $$\frac{d}{dt}\int_{MV} \rho b \, d\forall = \frac{d}{dt}\int_{CV} \rho b \, d\forall + \int_{CS} \rho b (\underline{u} - \underline{u}_s) \hat{n} \, dA$$

$\underline{u}_s$ – velocity of control surface

II. $$\frac{d}{dt}\int_{MV} \rho b \, d\forall = \int_{CV} \frac{\partial(\rho b)}{\partial t} \, d\forall + \int_{CS} \rho b (\underline{u} \cdot \hat{n}) \, dA$$

Leibnitz theorem :- $$\frac{d}{dt}\int_{\forall} (\rho b) \, d\forall = \int_{\partial\forall} \frac{\partial(\rho b)}{\partial t} \, d\forall + \int_{\partial\forall} \rho b (u_s \cdot \hat{n}) \, dA$$

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial \rho}{\partial x_j} = 0 \quad \Rightarrow \quad \frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0$$

$$\frac{D\rho}{Dt} + \rho(\underline{\nabla} \cdot \underline{u}) = 0$$

(1) Incompressible

* density of fluid element is constant

$$\boxed{\frac{D\rho}{Dt} = 0} \Rightarrow \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0 \Rightarrow \boxed{\underline{\nabla} \cdot \underline{u} = 0}$$

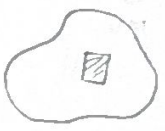
Balance of linear momentum.

Newton's law → particle

Euler law → continuum. ✓

Euler equation of motion

$$\frac{d}{dt}\left[\int_{MV} \rho \underline{u}\, d\forall\right] = \sum \underline{F}_{ext} = \underline{F}_{body} + \underline{F}_{surface}$$



$mass = \rho d\forall$

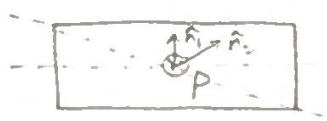
$dL = \rho u\, d\forall$

$$\frac{d}{dt}\int_{MV} \rho \underline{u}\, d\forall = \underbrace{\int_{\forall} \rho \underline{f}\, d\forall}_{\text{body}} + \underbrace{\int_{S} \underline{t}\, dA}_{\text{traction}}$$

$\underline{f}$ – body force per unit mass

$\underline{t}$ – traction force / area, stress vector

Cauchy stress principle



Body	Surface
- over a volume	- interaction of surfaces
- act at a distance	- over a surface
- long range	- very short range

1) At a given point, contact force depends on surface

$$\underline{t}(\underline{x} : d\underline{r}) \qquad d\underline{r} - \text{surface}$$

2) Traction depends on surface normal

$$\underline{t}(\underline{x} : d\underline{r}) = \underline{t}(\underline{x} : \hat{n})$$

3) Relation between traction and surface normal is linear

$$\underline{t} = \sigma^T \hat{n} \qquad t_i = n_j \sigma_{ji}$$

(σ ↳ stress)

In Hydrostatics, the force on the fluid element is normal to the surface. Thus, it is a special case of Cauchy (3).

Stress tensor, σ_{ij} (surface direction, force direction)

$$t_1 = n_j \sigma_{j1} = n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}$$

$t_2 =$

$t_3 =$

$$\Rightarrow \frac{d}{dt}\int_{MV} \rho u_i \, dV = \int_A \rho f_i \, dV + \int_S t_i \, dA$$

$$= \int_A \rho f_i \, dV + \int_S n_j \sigma_{ji} \, dA$$

$$= \int_A \rho f_i \, dV + \int_A \frac{\partial}{\partial x_j}(\sigma_{ji}) \, dV$$

$$= \int_A \left[\rho f_i + \frac{\partial}{\partial x_j}(\sigma_{ji})\right] dV$$

LHS

$$\frac{d}{dt}\int_{MV} \rho u_i \, dV = \int_{CV} \frac{\partial}{\partial t}(\rho u_i) \, dV + \int_{CS} \rho u_i (u_j n_j) \, dS$$

$$= \int_{CV} \frac{\partial}{\partial t}(\rho u_i) \, dV + \int_A \frac{\partial}{\partial x_j}(\rho u_i u_j) \, dV$$

$$= \int_A \left[\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j)\right] dV$$

Equating and using localisation theorem,

$$\underbrace{\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j}} = \rho f_i + \frac{\partial \sigma_{ji}}{\partial x_j}$$

$$\rho \frac{D(u_i)}{Dt} = \rho f_i + \frac{\partial \sigma_{ji}}{\partial x_j}$$

$$\rho \frac{D\underline{u}}{Dt} = \rho \underline{f} + \underline{\nabla} \cdot \underline{\underline{\sigma}} \quad \rightarrow \text{Cauchy Eqn of motion}$$

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Balance of angular momentum

$\sigma_{ij} = \sigma_{ji}$, $\underline{\underline{\sigma}} = \underline{\underline{\sigma}}^T$ [If any dipole is present in the fluid, this is not satisfied]

Balance of Total Energy

$$e_t = e + \frac{1}{2}u^2 + gz$$

$E_t = m e_t$

$$\left.\frac{dE_t}{dt}\right|_{sys} - \dot{Q} + \dot{W} = \frac{d}{dt}\int_{CV} \rho e_t \, d\forall$$

$$\dot{Q} = -\underbrace{\oint_{CS} \underline{q}\cdot\hat{n}\, dA}_{\text{Surface heat transfer}} + \underbrace{\int_{\forall} \dot{Q}\, d\forall}_{\text{volumetric heat transfer}}$$

Conduction

Chemical reaction

Radiation

$$\dot{W}_f = \int_{\forall} \rho f_i u_i + \int_S t_i u_i \, dA$$

$$= \int_{\forall} \rho f_i u_i + \int_S \sigma_{ji} n_j u_i \, dA$$

$$= \int_{\forall} \rho f_i u_i + \int_{\forall} \frac{\partial}{\partial x_j}(\sigma_{ji} u_i)\, dA$$

$$\dot{W}_s = \dot{W}_f + \dot{W}_{shaft} + \dot{W}_{other}$$

$$\dot{W} = \underline{F}\cdot\underline{u}$$

Similarly, $\dot{Q} = -\int_{CS} q_i n_i \, dS + \oint \dot{Q}_v$

$$= -\int_{\forall} \frac{\partial q_i}{\partial x_i}\, d\forall + \dot{Q}_v$$

Thus, $\cancel{\frac{d}{dt}(e_t)} = \int_{\forall}\left[-\frac{\partial q_i}{\partial x_j} + \rho f_i u_i + \frac{\partial}{\partial x_j}(\sigma_{ji} u_i)\right] d\forall = \frac{d}{dt}\int_{\forall} \rho e_t \, d\forall$

Applying RTT to RHS,

$$RHS = \cancel{\int_{\forall} \frac{\partial(\rho e_t)}{\partial t}\, d\forall} \int_{\forall} \frac{\partial(\rho e_t)}{\partial t} + \int_{\forall} \rho e_t (\underline{u}\cdot\hat{n})\, dA$$

By localisation theorem, (no field work)

$$\rho \frac{De_t}{Dt} = \rho f_i u_i + \frac{\partial(\sigma_{ji} u_i)}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \dot{Q}_v + \dot{W}_{other}$$

$$\rho \frac{De_t}{Dt} = \rho \underline{f}\cdot\underline{u} + \underline{\nabla}\cdot\underline{\underline{\sigma}}\cdot\underline{u} - \underline{\nabla}\cdot\underline{q}$$

Balance of mechanical energy

$u_i \times$ [momentum eqn]

$$u_i \rho \frac{Du_i}{Dt} = \rho f_i u_i + u_i \frac{\partial \sigma_{ji}}{\partial x_j}$$

$$\rho \frac{D\left(\frac{u_i u_i}{2}\right)}{Dt} = \rho f_i \cdot u_i + u_i \frac{\partial \sigma_{ji}}{\partial x_j}$$

$$\rho \frac{D}{Dt}\frac{|u|^2}{2} = \rho \underline{f} \cdot \underline{u} + \underline{u} \cdot \underline{\nabla} \cdot \underline{\underline{\sigma}}$$

Balance of Thermal energy

Total energy eqn. – Mechanical ener eqn.

$$\rho \frac{D\left(e + \frac{u_i u_i}{2}\right)}{Dt} = \text{- - - - chain rule on } \underline{\nabla} \cdot \underline{\underline{\sigma}} \cdot \underline{u}$$

$$\therefore \quad \rho \frac{De}{Dt} = \sigma_{ji} \frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_j}$$

We have,

$$\frac{\partial(\sigma_{ji} u_i)}{\partial x_j} = \sigma_{ij}\left(\frac{\partial u_i}{\partial x_j}\right) + u_i\left(\frac{\partial \sigma_{ji}}{\partial x_j}\right)$$

- $\sigma_{ij}\left(\frac{\partial u_i}{\partial x_j}\right)$: Stress × deformation rate ⇓ Increase internal energy
- $u_i\left(\frac{\partial \sigma_{ji}}{\partial x_j}\right)$: $u \times$ Force ⇓ Increase KE

Enthalpy

$$h = e + P/\rho \quad \Rightarrow \quad \rho \frac{Dh}{Dt} = \rho \frac{De}{Dt} + \rho \frac{D}{Dt}\left(\frac{P}{\rho}\right)$$

$$\Rightarrow \rho \frac{D}{Dt}\left(\frac{P}{\rho}\right) = \rho\left[\frac{1}{\rho}\frac{DP}{Dt} + P\left(\frac{-1}{\rho^2}\right)\frac{D\rho}{Dt}\right]$$

$$= \frac{DP}{Dt} + P\left(-\frac{1}{\rho}\right)\frac{D\rho}{Dt}$$

$$\Rightarrow \rho \frac{D}{Dt}\left(\frac{P}{\rho}\right) = \frac{DP}{Dt} + P\frac{\partial u_k}{\partial x_k}$$ ← From Mass conservation (incompressible).

$$\Rightarrow \text{Therefore} \quad \boxed{\rho \frac{Dh}{Dt} = \sigma_{ji}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \frac{DP}{Dt} + \rho\frac{\partial u_k}{\partial x_k}}$$

Stagnation enthalpy

$$\sigma_{ij} = \underbrace{-P\delta_{ij}}_{\text{at rest}} + \underbrace{\tau_{ij}}_{\text{moving}}$$

$$\rho \frac{Du_i}{Dt} = \rho f_i + \frac{\partial P}{\partial x_i} + \frac{\partial(\tau_{ji})}{\partial x_j}$$

$$\rho \frac{De_t}{Dt} = \rho f_i u_i - \frac{\partial P u_j}{\partial x_j} + \frac{\partial(\tau_{ji}u_i)}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

$$\rho \frac{De}{Dt} = -P\frac{\partial u_k}{\partial x_k} + \tau_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

Again from enthalpy. stagnation enthalpy

$$\rho \frac{Dh}{Dt} = \tau_{ij}\frac{\partial u_i}{\partial x_j} - \frac{\partial q_i}{\partial x_j} + \frac{DP}{Dt}.$$

$$\frac{D\left(h + \frac{u_i u_i}{2}\right)}{Dt} = \frac{\partial P}{\partial t} - \frac{\partial(\tau_{ij}u_i)}{\partial x_j} - \frac{\partial q_k}{\partial x_k} + \rho f_i u_i$$

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It is generally not possible to solve the above equations because number of unknowns is greater than number of equations.

Constitutive relation ★

- Non-directional
  - Equation of state
    - TEOS $f(P, \rho T) = 0$
    - CEOS $f(e, P, T) = 0$
      $f(h, PT) = 0$
- Directional
  - Transport equations
    - Mass
    - Momentum
    - Energy

Assumptions

* Equation of state (Thermally perfect gas)

- Intermolecular forces are neglected.

Thermal EOS :- $p = \rho R T$

Caloric EOS :- $e = f(T)$ $\quad de = c_v(T)\,dT$

$h = f(T)$ $\quad dh = c_p(T)\,dT$

calorically perfect

TEOS :- $p = \rho R T$

CEOS :- $e = c_v T$ ~~= const~~ $\quad c_v, c_p$ = constant

$h = c_p T$ ~~= const~~

Thermally non-perfect.

$$\rho \frac{Dh}{Dt} = c_p \frac{DT}{Dt} + \frac{(1-\beta T)}{\rho} \frac{D\rho}{Dt}$$

Thermally perfect

$$\rho \frac{Dh}{Dt} = c_p \frac{DT}{Dt}$$

β - Coefficient of thermal expansion

$$= -\frac{1}{\rho} \left.\frac{d\rho}{dT}\right|_p$$

Newtonian Fluid. (Momentum eqn.)

1) When fluid is at rest ⇒ stress only hydrostatic.

$$\sigma_{ij} = \underbrace{-p\delta_{ij}}_{\text{Fluid static}} + \underbrace{\tau_{ij}}_{\text{Fluid-dynamic}} \longrightarrow \text{stress in a moving fluid.}$$ ← (deviatoric)

2) τ_{ij} is linear with $\frac{\partial u_i}{\partial x_j}$. $\quad \tau_{ij} = \underset{\text{4th order}}{\alpha_{ijmn}} \frac{\partial u_m}{\partial x_n} \frac{\partial u_n}{\partial x_m}$ ✗

3) τ_{ij} (stress) should not depend on rigid body motion (rotation). [Galelian Invariance]

$$\tau_{ij} = \alpha_{ijmn} (S_{mn} + \overset{0}{R_{mn}})$$

$$\tau_{ij} = \alpha_{ijmn} S_{mn}$$

4) For isotropic fluid, constants in δ_{ijmn} reduces from 81 to 2.

$$\sigma_{ij} = -P\delta_{ij} + \lambda\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ij} + 2\mu S_{ij}$$

(S_{mn})

μ - dynamic coefficient of viscosity

λ - second " " "

$$\sigma_{ii} = -P\delta_{ii} + \lambda\left(\frac{\partial u_k}{\partial x_k}\right)\delta_{ii} + \cancel{2\mu\left(\frac{\partial u_k}{\partial x_k}\right)} \; 2\mu S_{ii}$$

$$= -3P + 3\lambda\left(\frac{\partial u_k}{\partial x_k}\right) + 2\mu\left(\frac{\partial u_k}{\partial x_k}\right)$$

$$\sigma_{ii} = -3P + (3\lambda + 2\mu)\frac{\partial u_k}{\partial x_k}.$$

Average normal stress $\Rightarrow$ Mechanical pressure.

$$-\bar{P}_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = -P + \left(\lambda + \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}$$

Thermodynamic pressure

$$P - \bar{P}_m = \left(\lambda + \frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k}$$

Mechanical pressure

Thermodynamic pressure ?

For incompressible flow, $\frac{\partial u_k}{\partial x_k} = 0 \Rightarrow \boxed{P = \bar{P}_m}$

compressible flow, $P - \bar{P}_m = \left(\lambda + \frac{2}{3}\mu\right)\nabla\cdot\underline{u}$

$$= \mu_b(\nabla\cdot\underline{u})$$

μ_b ↑ Bulk viscosity

Stoke's hypothesis

$\boxed{\mu_b = 0}$ $\boxed{\lambda = -\frac{2}{3}\mu}$.

⁂ Not applicable → when normal stress >> Shear stress.

Constitute relation for Newtonian fluids:-

$$\sigma_{ij} = -p\delta_{ij} + \left(-\frac{2}{3}\mu\right)\frac{\partial u_k}{\partial x_k} + 2\mu S_{ij}$$

Fourier Law of Heat conduction (thermal energy)

q_i linear with $\frac{\partial T}{\partial x_i}$

For isotropic material, $K_{ij} = KI$

$$q_i = -K_{ij}\frac{\partial T}{\partial x_j}$$

$$\boxed{q_i = K\frac{\partial T}{\partial x_i}}$$

Substitute these constitute relations into the equations derived previously

$$\frac{D\rho}{Dt} + \rho\frac{\partial u_k}{\partial x_k} = 0$$

Substituting constitutive relation into cauchy's equation gives Navier-Stokes equation

$$\rho\frac{Du_i}{Dt} = \rho f_i + \frac{\partial \sigma_{ji}}{\partial x_j}$$

$$= \rho f_i + \frac{\partial}{\partial x_j}\left[-p\delta_{ij} + \lambda^{\delta_{ij}}\frac{\partial u_k}{\partial x_k} + 2\mu S_{ij}\right]$$

$$\rho\frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\lambda\frac{\partial u_k}{\partial x_k}\delta_{ij} + 2\mu S_{ij}\right] \qquad \mu = f(p,T) \approx f(T)$$

Assuming μ = const. & $\lambda = -\frac{2}{3}\mu$

$$\rho\frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \cancel{\frac{\partial}{\partial x_j}\left(-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\right)} - \frac{2}{3}\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) + \frac{2}{3}\mu\frac{\partial}{\partial x_j}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

N-S momentum eqn.

$$\rho\frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)$$

Assume incompressible, $\rho\frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}$

Assume eq. inviscid $\rho\frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i}$

Next to read

non-uniform Classical Kinematics

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Similarly, substitute constitute relations into thermal energy eqn.

$$\rho \frac{De}{Dt} = -\frac{\partial q_j}{\partial x_j} + \sigma_{ij} \frac{\partial u_i}{\partial x_j} = -\frac{\partial q_j}{\partial x_j} - p \frac{\partial u_k}{\partial x_k} + \tau_{ij} \frac{\partial u_i}{\partial x_j}$$

Here, $\tau_{ij} \frac{\partial u_i}{\partial x_j} = \tau_{ij} [S_{ij} + R_{ij}] = \tau_{ij} S_{ij} + 0$

Substituting τ_{ij} from previous,

$$\tau_{ij} \frac{\partial u_i}{\partial x_j} = \left[\lambda \frac{\partial u_k}{\partial x_k} \delta_{ij} + 2\mu S_{ij}\right] S_{ij}$$

$$= \lambda \frac{\partial u_k}{\partial x_k} S_{ii} + 2\mu S_{ij} S_{ij}$$

$$= \underbrace{\lambda \frac{\partial u_k}{\partial x_k} \frac{\partial u_k}{\partial x_k} + 2\mu S_{ij} S_{ij}}$$

$\Phi \rightarrow$ dissipation function

$\varepsilon = \frac{\Phi}{\rho}$ dissipation rate per unit mass

$\boxed{\Phi > 0}$

$$\Rightarrow \rho \frac{De}{Dt} = -\frac{\partial q_j}{\partial x_j} - p \frac{\partial u_k}{\partial x_k} + \Phi$$

Similarly, substituting in mechanical energy.

$$\rho \frac{D\left(\frac{u_i u_i}{2}\right)}{Dt} = u_i \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i u_i = \frac{\partial}{\partial x_j}(\sigma_{ij} u_i) - \sigma_{ij} \frac{\partial u_i}{\partial x_j} + \rho f_i u_i$$

$$= \frac{\partial (\sigma_{ij} u_i)}{\partial x_j} - \left[-p \frac{\partial u_k}{\partial x_k} + \Phi\right]$$

Here, it can be seen that Φ appears as +ve in themal eqn. and -ve in the mechanical energy.

Here, $\frac{\partial}{\partial x_j}(\sigma_{ij} u_i) = \underbrace{u_i \frac{\partial \sigma_{ij}}{\partial x_j}}_{\text{Increases KE}} + \underbrace{\sigma_{ij} \frac{\partial u_i}{\partial x_j}}_{\text{Increases internal energy}} = \left[-u_j \frac{\partial p}{\partial x_j} + u_i \frac{\partial \tau_{ij}}{\partial x_j}\right] + \left[-p \frac{\partial u_k}{\partial x_k} + \Phi\right]$

Reversible — Irreversible

Entropy eqn

$$T\frac{Ds}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho}\frac{DP}{Dt} \quad \Rightarrow \quad \rho\frac{Ds}{Dt} = \frac{1}{T}\rho\frac{Dh}{Dt} - \frac{1}{T}\frac{DP}{Dt}$$

$$\Rightarrow \quad \rho\frac{Ds}{Dt} = \frac{1}{T}\left[\rho\frac{De}{Dt} + \frac{DP}{Dt} + P\frac{\partial u_k}{\partial x_k}\right] - \frac{1}{T}\frac{\partial P}{Dt}$$

$$= \frac{1}{T}\left[\frac{-\partial q_j}{\partial x_j} - P\frac{\partial u_k}{\partial x_k} + \Phi + P\frac{\partial u_k}{\partial x_k}\right]$$

$$\rho\frac{Ds}{Dt} = -\frac{1}{T}\frac{\partial q_j}{\partial x_j} + \frac{\Phi}{T}$$

$$= -\left[\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) + \frac{q_j}{T^2}\frac{\partial T}{\partial x_j}\right] + \frac{\Phi}{T}$$

$$= -\frac{\partial}{\partial x_j}\left(\frac{q_j}{T}\right) - \frac{q_j}{T^2}\frac{\partial T}{\partial x_j} + \frac{\Phi}{T}$$

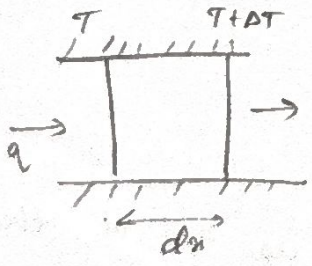
$$\boxed{\rho\frac{Ds}{Dt} = \frac{\partial}{\partial x_j}\left(k\frac{\partial T}{\partial x_j}\right) + \frac{k}{T^2}\left(\frac{\partial T}{\partial x_j}\right)\left(\frac{\partial T}{\partial x_j}\right) + \frac{\Phi}{T}}$$

Reversible $\quad >0 (k>0) \quad >0 (\mu>0)$

Analysis

$$\int_V \rho\frac{Ds}{Dt}\,dV = \int_S \dot{h}\cdot\left(\frac{k}{T}\cdot\frac{\partial T}{\partial x_j}\right)dA + \int_V\left[\frac{k}{T^2}\left(\frac{\partial T}{\partial x_j}\right)\left(\frac{\partial T}{\partial x_j}\right) + \frac{\Phi}{T}\right]dV$$

Consider



$$S_{in} = \frac{q}{T}\,dy\,dz$$

$$S_{out} = \frac{q}{T+\Delta T}\,dy\,dz$$

$$= \frac{q}{\left(T + \frac{dT}{dx}dx\right)}\,dy\,dz = \frac{q}{T}\left(1 + \frac{1}{T}\frac{dT}{dx}dx\right)^{-1}$$

$$= \frac{q}{T}\left(1 - \frac{1}{T}\frac{dT}{dx}dx\right)$$

$$\therefore \quad \frac{S_{out} - S_{in}}{V} = \frac{\frac{q}{T}\left(-\frac{1}{T}\frac{dT}{dx}dx\,dy\,dz\right)}{V}$$

$$= -\frac{q}{T}\left(\frac{1}{T}\frac{dT}{dx}\right) = \frac{k}{T^2}\left(\frac{\partial T}{\partial x}\right)^2$$

>0

[Eg. for how heat transfer generates entropy]

* Thermodynamics can be only applied for equilibrium flow. But, our fluid flow has gradient → so cannot apply

* Workaround → ~~#~~ local equilibrium theory → considers first disturbance creates new equilibrium very quickly (relaxation time).

Governing Equations

	No. of unknowns	No of eqn
1) Mass conservation	$\rho, \underline{u}$	1
2) Balance of momentum	$\rho, \underline{u}, P, \mu$ [4]	+1 = 2
3) Energy	$\rho, \underline{u}, P, \mu, h, T, K$ [7]	+1 = 3
		+1 = 4
4) Thermal EOS		+1 = 5
5) Caloric EOS		+1 = 6
6) $\mu(P,T) \approx \mu(T)$ $k(P,T) \approx k(T)$		+2 = 7

Thus, 7 equations and 7 unknowns can be solved. But, we try to simplify problem to reduce the number of unknowns to make it easy

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Mass & momentum

$\rho, \underline{u}, p \rightarrow$ 2 eqns.

$\rightsquigarrow$ $p(\rho, T) \Rightarrow$ Energy eqn. is needed

$\Rightarrow$ $p(\rho) \Rightarrow$ Barotropic ~~constant~~ flow.

~~Eg~~ $p/\rho^\gamma = c \rightarrow$ isentropic flow

$p/\rho = c \rightarrow$ isothermal flow

$\rho = c$

Now, from momentum equation,

$$\frac{d}{dt}\int_{CV} \rho u_i dV = -\int p\hat{n}\,dA + \int \underline{t}\,dA + \int \rho f\,dV$$

$$\int_{CV} \frac{\partial(\rho \underline{u})}{\partial t} dV + \int_{CS} \rho \underline{u}\,(\underbrace{\underline{u}\cdot\hat{n}}_{\text{scalar already}})\,dA$$

HW
1) Write all the eqn. in component form.

Writing in components of x, y, z;

$$\int_{CV} \frac{\partial(\rho u)}{\partial t} dx\,dy\,dz + \int_{CS} \rho u(\underline{u}\hat{n})\,dA = -\int p\,\hat{n}\cdot\hat{e}_x\,dA + \int \underline{t}\cdot\hat{e}_x\,dA + \int \rho \underline{f}\cdot\hat{e}_x\,dV$$

(x-component)

Writing the three conservation equations in compact form (system of eqn).

$$\frac{\partial B}{\partial t} + \frac{\partial F}{\partial x} = Q \qquad \text{[Conservative form]}$$

$$B = \begin{Bmatrix} \rho \\ \rho \underline{u} \\ \rho e_t \end{Bmatrix} \qquad F = \begin{bmatrix} \rho u_i \\ \rho u_i u_j + p\delta_{ij} - \tau_{ji} \\ (\rho e_t + p)u_j - \tau_{kj}u_k - k\frac{\partial T}{\partial x_j} \end{bmatrix} \qquad Q = \begin{bmatrix} 0 \\ \rho f_i \\ 0 \end{bmatrix}$$

BOUNDARY CONDITIONS

I) Velocity

1) ⊥le to surface
   - no-slip ← relative velocity b/w fluid and surface adjacent to it should be zero.
     $(u_{t,w} - u_{t,w}) = 0$
     ($u_{t,w}$: Fluid tangent; $u_{t,w}$: Wall)
   - slip

As Nusselt number goes above 1, the slip in fluid flow tends to appear. For low Nusselt no. values, the flow is no-slip (Both flow is valid continuum mechanics).

2) ⊥le to surface
   - no-penetration ← normal component of velocity should be zero.
   - $u_n \neq 0$ ← suction / Blowing

II) Temperature

1) Dirichlet, T

2) Neuman (q₀), $\frac{\partial T}{\partial x}$

3) Mixed

Special form of Governing Equation

To reduce the number of GE by assumptions

1) Bernoulli eqn.

2) Croco's eqn

3) Circulation eqn

4) Vorticity eqn

Bernoulli's Eqn

Assumption

1. Inviscid
2. Conservative body force (Body force can be written as a potential)
3. Barotropic flow.

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = \frac{-1}{\rho}\nabla P + \underline{f} \qquad \text{——} (1)$$

where, $\underline{f} = -\nabla G$ $\quad G \rightarrow$ potential

$= -\nabla(gz)$

Rewriting eq (1), $(\underline{u}\cdot\nabla)\underline{u}$

$$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla\left(\frac{|u|^2}{2}\right) - (\underline{u}\times\underline{\omega})} = -\frac{1}{\rho}\nabla P - \nabla(gz) \qquad \text{——} (2)$$

For Barotropic : $\frac{1}{\rho}\nabla P = \nabla\int\frac{dP}{\rho}$ (Kundu, Pg 111) eq 4.68

Thus,

$$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla}\left(\int\frac{dP}{\rho} + \frac{|\underline{u}|^2}{2} + gz\right) = \underline{u}\times\underline{\omega} \qquad \text{——} (3)$$

($\underline{\omega}$: vorticity (omega))

$$\frac{\partial \underline{u}}{\partial t} + \underline{\nabla}B = \underline{u}\times\underline{\omega} \qquad B = \int\frac{dP}{\rho} + \frac{|\underline{u}|^2}{2} + gz \qquad \text{——} (4)$$

For steady flow, $\frac{\partial \underline{u}}{\partial t} = 0$.

$$\therefore \quad \boxed{\underline{\nabla}\cdot B = \underline{u}\times\underline{\omega}} \qquad \text{——} (5)$$

For a streamline e_s,

$$e_s\cdot\nabla\cdot B = e_s\cdot(\underline{u}\times\underline{\omega})$$

$$\frac{dB}{ds} = 0 \qquad \therefore B \text{ is constant along streamline}$$

For irrotational flow, it is constant for all streamlines

Deriving Bernoulli eqn from energy equation,

$$\rho \frac{Dh_t}{Dt} = \frac{\partial p}{\partial t} + \frac{\partial (\tau_{ij} u_i)}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho f_i u_i \qquad \text{——⑥}$$

Assumptions

1. Inviscid
2. Adiabatic
3. Conservative body force
4. Steady.

Analysis

Here, h_0 is constant for a streamline. But the accompanying assumptions are 4.

From eq ⑥,

$$\rho \underline{u} \cdot \nabla\left(h + \frac{|u|^2}{2}\right) = -\rho u \nabla(gz)$$

Many terms cancel out.

$$\rho \underline{u} \cdot \underbrace{\nabla\left(h + \frac{|u|^2}{2} + gz\right)}_{\perp \text{ to } \underline{u}} = 0 \qquad \text{——⑦}$$

Thus, $\underbrace{h + \frac{|u|^2}{2} + gz}_{h_0} =$ constant along the streamline

From Gibbs equation

$$T\cancel{ds}^{\,0} = dh - \frac{dp}{\rho}$$

$$dh = \frac{dp}{\rho} \Rightarrow h = \int \frac{dp}{\rho}.$$

∴ Eq ⑦ becomes

$$\int \frac{dp}{\rho} + \frac{|u|^2}{2} + gz = \text{constant along streamline.}$$

Because flow is 1D and h_0 can be constant only for adiabatic flow.

valid for integral sense.

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Crocco's Eqn.

- Connects thermodynamics with kinematics

$$Tds = dh - \frac{dp}{\rho}$$

$$T\nabla s = \nabla h - \frac{\nabla p}{\rho} = \nabla h + \left[\frac{\partial \underline{u}}{\partial t} + \frac{\nabla |\underline{u}|^2}{2} - \underline{u} \times \underline{\omega}\right] + \nabla(gz)\right] \quad \text{——— (1)}$$

Assumption

1. Inviscid
2. Conservative body force

$$\therefore \quad T\nabla s = \nabla h_o - \underline{u} \times \underline{\omega} + \frac{\partial \underline{u}}{\partial t} \quad \text{——— (2)}$$

For steady flow,

$$\boxed{T\nabla s + \underline{u} \times \underline{\omega} = \nabla h_o} \quad \text{——— (3)}$$

If h_o is constant, $\nabla h_o = 0$.

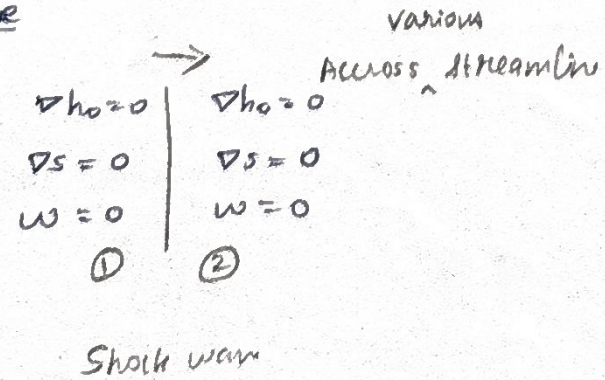
$$\Rightarrow \quad T\nabla s + \underline{u} \times \underline{\omega} = 0$$

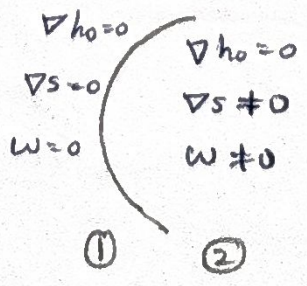
For isentropic flow, $\nabla s = 0$.

$$\Rightarrow \quad \underline{u} \times \underline{\omega} = 0 \qquad [\text{For 2D flow, } \underline{\omega} = 0]$$

Thus, for isentropic steady flow, it is irrotational.

Example





Circulation Eqn.

Rate of change of circulation of a material curve

Material curve - Curve made up of same ~~a~~ fluid particle.

$$\Gamma = \oint u_i dx_i \quad , \quad \frac{D\Gamma}{Dt} = \frac{D}{Dt}\left[\oint u_i dx_i\right] = \oint \frac{D}{Dt}(u_i dx_i)$$

dx_i → infinitesimal change in length

$$\therefore \frac{D\Gamma}{Dt} = \oint \left[u_i \frac{D(dx_i)}{Dt} + dx_i \frac{Du_i}{Dt}\right]$$

$$\oint u_i \frac{D(dx_i)}{Dt} = \oint u_i du_i$$

$$= \oint d\left(\frac{u_i u_i}{2}\right) = 0$$

Thus, $$\frac{D\Gamma}{Dt} = \oint \left(-\frac{\nabla P}{\rho} + F_v + F_B^{\text{Body}}\right) dx_i$$ → Pressure torque, Viscous torque, Body torque

Using Stoke's theorem will introduce the cross product.

When there is no torque,

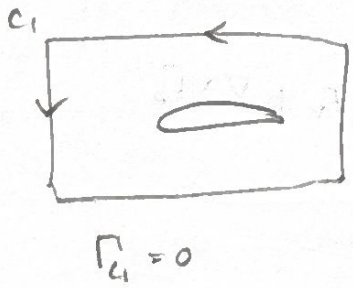
$$\boxed{\frac{D\Gamma}{Dt} = 0}$$ Kevin Circulation Theorem

Assumption,

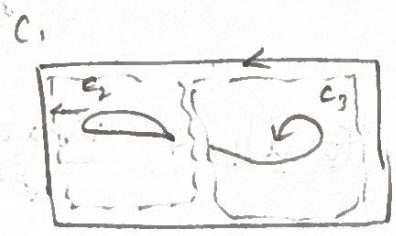
1) Barotropic
2) Conservative body force
3) Inviscid.

For airfoil

Kelvin circulation Theorem predicts some of Lift.



$\Gamma_{C_1} = 0$



$\Gamma_{C_1} = 0$

$$\Gamma_{C_1} = \Gamma_{C_2}^{(-ve)} + \Gamma_{C_3}^{(+ve)}$$

Thus, the curves C_2 & C_3 cut through the shear layer of wake.

Starting vortex is produced by viscous effect. This leads to viscous torque due to choice of C_2 and C_3.

Vorticity Transport Eqn

$$\underline{\omega} = \nabla \times \underline{u}$$

$$= \nabla \times (\text{momentum eqn})$$

$$= \nabla \times \left[\frac{\partial \underline{u}}{\partial t} + \nabla \frac{|u|^2}{2} - \underline{u} \times \underline{\omega} = \frac{-\nabla P}{\rho} + F_v + F_b = 0 \right] \quad \text{——①}$$

Vector identities

1. $\nabla \times \nabla \phi = 0$
2. $\nabla \times (\phi A) = \phi (\nabla \times A) + \nabla \phi \times A$

Thus, eqn ① becomes

$$\nabla \times \left(\frac{1}{\rho} \nabla P \right) = \frac{1}{\rho} \nabla \times \nabla P^{0} + \nabla \left(\frac{1}{\rho} \right) \times \nabla P = \frac{-\nabla \rho \times \nabla P}{\rho^2}$$

3. $\nabla \times A \times B = (\nabla \cdot B) A + (B \cdot \nabla) A - (\nabla \cdot A) B - (A \cdot \nabla) B$

$$\nabla \times \underline{u} \times \underline{\omega} = (\nabla \cdot \underline{\omega})^{0} \cdot \underline{u} + (\underline{\omega} \cdot \nabla) u - (\nabla \cdot u) \omega - (u \cdot \nabla) \omega$$

$$\nabla \times \frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{\omega}}{\partial t}$$

$$\therefore \quad \frac{\partial \underline{\omega}}{\partial t} + (\underline{u} \cdot \nabla) \underline{\omega} = -(\nabla \cdot u) \underline{\omega} + (\underline{\omega} \cdot \nabla) \underline{u} + \frac{\nabla \rho \times \nabla P}{\rho^2} + \nabla \times F_v + \nabla \times F_b$$

$$\frac{D \underline{\omega}}{Dt} = \underbrace{(\underline{\omega} \cdot \nabla) \underline{u} - (\nabla \cdot \underline{u}) \underline{\omega}}_{\text{Kinematics}} + \underbrace{\frac{\nabla \rho \times \nabla P}{\rho^2} + \nabla \times F_v + \nabla \times f_b}_{\text{Torques}}$$

All torques are zero. — Incomp. flow

$$\frac{D \underline{\omega}}{Dt} = (\underline{\omega} \cdot \nabla) u - (\nabla \cdot \underline{u}) \underline{\omega} \qquad \frac{D \underline{\omega}}{Dt} = (\underline{\omega} \cdot \nabla) \underline{u} \quad [=0 \text{ for 2D}]$$

Unlike Circulation eqn, here even with no torque for incompressible flow, the flow vorticity still changes

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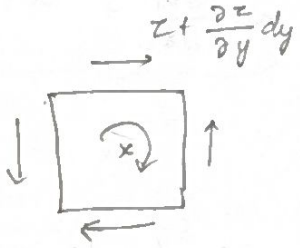
Incompressible viscous with conservative body force,

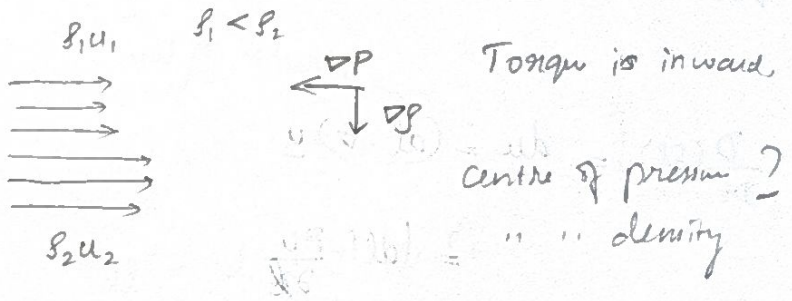
$$\frac{D\underline{u}}{Dt} = -(\underline{u}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{u}$$

Types of torque

1. Viscous torque : $\nabla \times F_v$



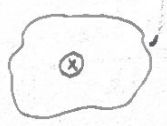
2. Baroclinic torque : $\dfrac{\nabla\rho \times \nabla P}{\rho^2}$



Torque is inward,

centre of pressure ? ,, ,, density

3. Body force torque : $\nabla \times F_b$



This force acts at CG, no torque

- If F_b is conservative $\Rightarrow$ $\nabla \times F_b = 0$

- If F_b is non conservative $\Rightarrow$

Eg: Coriolis force $(-2\underline{\Omega} \times \underline{u})$

Vorticity change and angular momentum change

- Vorticity is measure of angular velocity.
- Torque is zero → Angular momentum constant

Angular momentum, $L = \underline{I}\,\underline{\Omega} = \underline{I}\,\tfrac{1}{2}\underline{\omega}$

$$\frac{D}{Dt}\left(\tfrac{1}{2} I\omega\right) = 0 \qquad (\because \text{No torque})$$

Here, during fluid flow, the moment of inertia (I) changes. To satisfy the above condition, ω changes.

Relation between vorticity line & material line

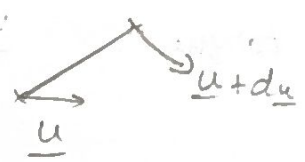
Material line – Line consists of same particle.

Vorticity line – Line tangent to vorticity vector.

$$\frac{D\omega_1}{Dt} = (\underline{\omega}\cdot\nabla)\underline{u}_1 = \underbrace{\omega_1 \frac{\partial u_1}{\partial x_1}}_{\text{Stretching}} + \underbrace{\omega_2 \frac{\partial u_1}{\partial x_2} + \omega_3 \frac{\partial u_1}{\partial x_3}}_{\text{Tilting}}$$

we want to know how the material line changes.

$$\frac{D}{Dt}(\delta l) = du = (\underbrace{dl}_{\text{Direction of line}}\cdot\nabla)\underline{u}$$

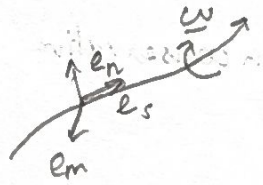


$$= |dl| \frac{\partial u}{\partial l}$$

$$\boxed{\frac{1}{|\delta l|}\frac{D(\delta l)}{Dt} = \frac{\partial u}{\partial l}}$$ (Behaviour of material line)

Now check the vorticity line.

$$\underline{\omega} = |\omega_s|\,\hat{e}_s$$



$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega}\cdot\nabla)\underline{u} = |\omega_s|\frac{\partial \underline{u}}{\partial s}$$ (No torque)

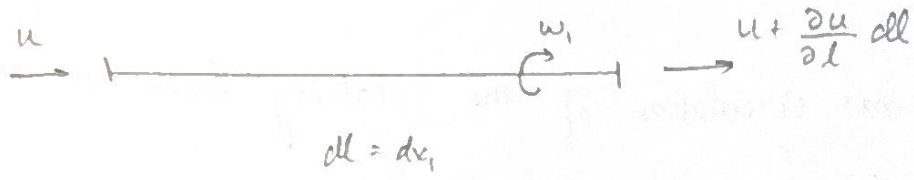
$$\boxed{\frac{1}{|\omega_s|}\frac{D\underline{\omega}}{Dt} = \frac{\partial u}{\partial s}}$$ (Behaviour of vorticity line)

* Comparing both these equations, it is inferred that behaviour of material line is same as vorticity line (assumption of no torque).

* Vorticity line is frozen in the flow. (the same way as fluid flow)

Vortex stretching

@ $t = 0$



$dl = dx_1$

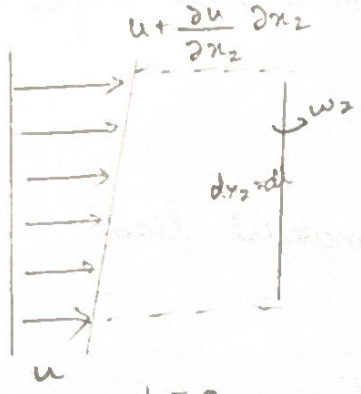
@ $t \geq 0$

$$dl' = dl + \frac{\partial u}{\partial l} dl\, dt$$

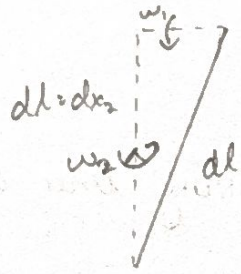
$$\frac{D(\delta l)}{Dt} = \frac{\partial u}{\partial l} dl$$

$$\frac{D\omega_1}{Dt} = \frac{\partial u}{\partial l} \omega_1 \left(= \omega_1 \frac{\partial u}{\partial x_1} \right)$$ → stretching.

Vortex tilting



$$dx_1 = \frac{\partial u_1}{\partial x_2} dx_2\, dt$$



$$dl' = dx_1 \hat{e}_1 + dx_2 \hat{e}_2$$

$$\frac{D(\delta l)}{Dt} = \frac{(dl' - dl)}{dt}$$ ← (Here, dx_2 component is cancelled)

$$\frac{D(\delta l)}{Dt} = \frac{\partial u_1}{\partial x_2} dx_2$$

(By analogy to material line)

Similarly, $\boxed{\frac{D\omega_1}{Dt} = \omega_2 \frac{\partial u_1}{\partial x_2}}$ ω_1 is created due to

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In 2D, $\frac{D\underline{\omega}}{Dt} = 0.$ ∴ Flow velocity is $u(x, y)$, $\omega_3 \hat{e}_3 = \underline{\omega}$.

$$(\underline{\omega} \cdot \nabla) u_1 = \omega_1 \frac{\partial u_1}{\partial x_1} + \omega_2 \frac{\partial u_1}{\partial x_2} + \omega_3 \frac{\partial u_1}{\partial x_3}$$ (the first and last terms are 0)

2D viscous flow

$$\frac{D\underline{\omega}}{Dt} = \nu \nabla^2 \underline{\omega}$$

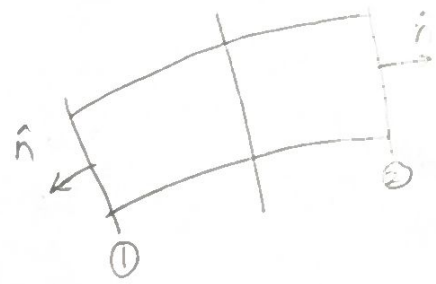
Vorticity theorems (Helmholtz)

1) The strength (or) circulation of the vorticity tube is constant at every cross section at a given time.

$$\nabla \cdot \underline{\omega} = 0 \qquad \left[\nabla \cdot (\nabla \times \underline{u}) = 0\right]$$

But $\nabla \cdot \underline{\omega} = \oint_A \underline{\omega} \cdot \hat{n}\, dA = 0$

$$\Gamma = \omega_1 A_1 = \omega_2 A_2$$



$\lim_{\forall \to 0} \int_{\forall} \nabla \cdot \underline{\omega}\, d\forall = [\nabla \cdot \underline{\omega}]\forall$

$\nabla \cdot \underline{\omega} = \frac{1}{\forall}$

$\lim_{\forall \to 0} \frac{1}{\forall} \oint_A \underline{\omega} \cdot \hat{n}\, dA = 0$

Consequence ⇒

- Vorticity line wont end in fluid. It either ends in a solid surface or loop itself. (Vortex rings)

2) Vorticity line move with the fluid (or) vorticity lines are material lines.

Assumption → No torque

3) Strength (or) circulation of the vorticity tube is invariant with time.

Assumption → No torque $\left[\frac{D\Gamma}{Dt} = 0\right]$

The vorticity eqn. is easy to interpret flow property. In Navier Stokes, the pressure term is diff.

In incompressible flow pressure is non-local. It is dependent on the velocity field of the whole flow.

In compressible flow, pressure is local.

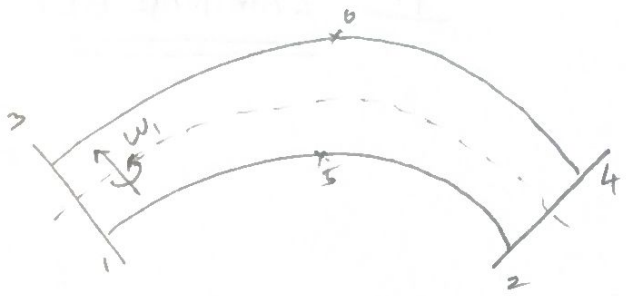
Secondary flow in a bend.

- A flow is generated in the fluid flow.
- ~~Usually~~ because of the vorticity of fluid because of the bend.

Using Streamline force theory.

$P_5 < P_6$

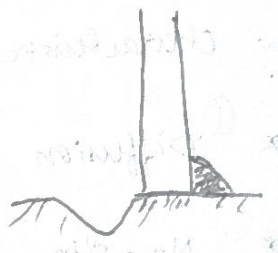
$\Rightarrow$ $u_5 > u_6$



Shapiro video.

H.W

what happens to w_1 at exit. for symmetric bend.



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II LAMINAR BLT

Viscous flow

- Transport properties are not zero (viscosity, conductivity, mass diffusivity)

New characteristics introduced in viscous flow?

* ① Diffusion (isotropic)
* No-slip
* Laminar to turbulent transition.

other transport phenomena
② - advection } classical
③ - wave motion

(Transport properties are proportional to gradient)

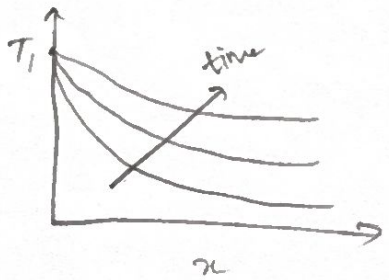
$q \propto \nabla T$

$\tau \propto \nabla u$

- Transport phenomena exists in gradients.

Diffusion is isotropic

Then Temperature

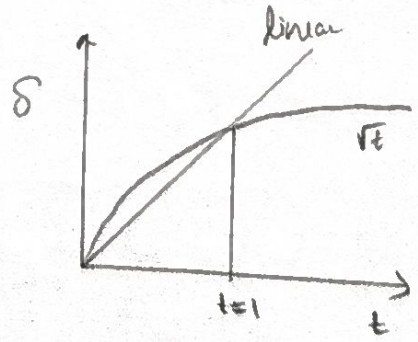


diffusive length

$(\delta) = \sqrt{\text{Diffusivity} \times \text{time}}$

Viscous phenomena is dictated by diffusivity ($\nu = \frac{\mu}{\rho}$ is momentum diffusivity)

$\delta < \sqrt{t}$



$K = \frac{k}{\rho C_p}$ (Thermal)

	μ ($\times 10^{-6}$)	ν
Air	18.18	15.3
Water	114.0	1.14
Mercury	1580	0.12

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Viscous

1) Diffusion
2) No-slip
3) Laminar → Turbulent

① Diffusion is an isotropic property and occurs wherever the gradient is present. It tries to nullify the gradient.

$$\text{Reynolds number} = \frac{\text{advection}}{\text{diffusion (viscous)}}$$

$$Pe_{th} = \frac{\text{advection}}{\text{Thermal diffusion}} \quad \text{(Peclet)}$$

$$Pr = \frac{\mu c_p}{K} = \frac{\nu}{\alpha} \quad \text{(Prandtl)} \qquad \frac{\text{Viscous}}{\text{Thermal}}$$

$$Sc = \frac{\nu}{D} \quad \text{(Schmidt)}$$

$$Le = \frac{\alpha}{D} \quad \text{(Lewis)}$$

Now, $Pe_{thermal} = Re \cdot Pr$

$Pe_{mass} = Re \cdot Sc$

② No slip

- Here, a new type of drag is introduced, friction drag.
- Aerodynamic heating / viscous heating.
- Flow seperation.

Analytical solution of Navier Stokes Equation

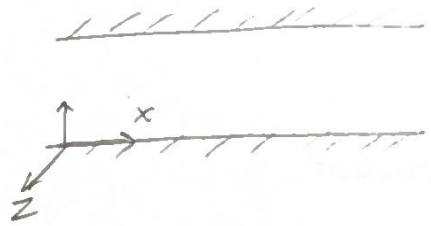
* Navier-Stokes is ~~analytical~~ non-linear (because of advection)

Advection acceleration

* By making advection to be zero, the N-S eqn. becomes linear. ($u \cdot \nabla u = 0$).

Eg - Fully developed
Pipeflow

Fully developed channel flow (Linear solution)



Assumptions
- Steady
- Fully developed $\frac{\partial}{\partial x}() = 0$
- Incompressible

For simplicity, take 2-D flow-

$$\cancel{\frac{\partial u}{\partial x}}^{0} + \frac{\partial v}{\partial y} = 0 \quad \text{(Incompressible)}$$

$\frac{\partial v}{\partial y} = 0 \rightarrow$ v is independent of y.

$\rightarrow v = v(x)$

Boundary condition,

@ $y=0$, $v=0$ (Boundary layer)

$\Rightarrow v = 0$ everywhere

$\Rightarrow \frac{\partial p}{\partial y} = 0$ (from streamline theory)

Solutions are linear. Thus, superposition principle is applicable

-X- Exam

HW

Pipe flow - Pressure driven

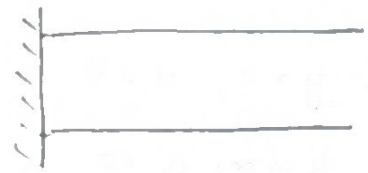
channel flow - Pressure driven +ve, -ve
- shear driven
- P+shear driven

* Profiles
$v(y)$, $\tau(y)$, u_{max}, u_{avg}
C_f

Similarity solution

Self similar solution certain flow can have this

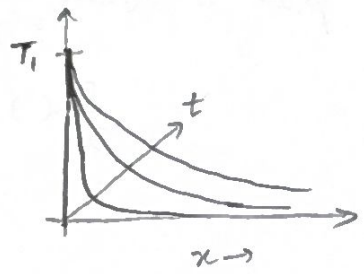
→ In bar heating problem, the temperature keeps changing in all x-positions over time. Thus, the "solution" must be stored or memorized for very infinite time.



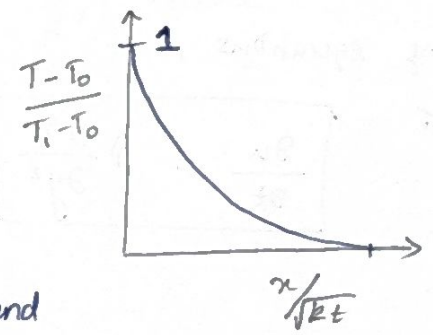
@ $t<0$ $T=T_0$

@ $t>0$ $T(x=0)=T_1$

→ By non-dimensionalizing / scaling, all the solution curve collapses into a single curve



→ Self similar solutions are very important in analytical and numerical calculation.

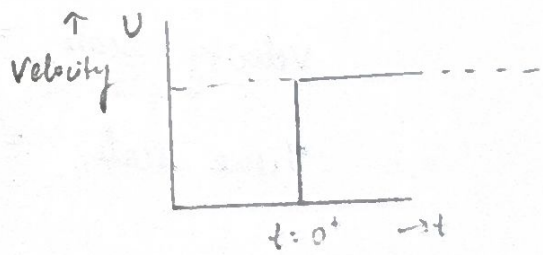


When can we expect self-similar solution?

→ If the problem does not have length scale and time scale ~~or both~~.

Impulsively started plate (Flat plate) (Stroke's First Problem)

Consider infinite plate exposed to atmosphere. An impulse is acted on this plate.



Assumption.

1. Incompressible flow, constant properties (K & u)
2. 2D flow $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$

Governing equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial y}+\nu\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right]$$

Boundary condition

$t < 0, \quad u = 0$

$t > 0, \quad y = 0, \quad u = U$

$y \to \infty, \quad u = 0.$

Because plate is infinite in x-direction,

Assumed absence of movement of plate.

$$\frac{\partial}{\partial x}(\,) = 0 \Rightarrow \boxed{v = 0} \text{ everywhere}$$

$$\boxed{\frac{\partial P}{\partial y} = 0}$$

Simplifying equations :-

~~$\frac{\partial u}{\partial t}$~~

$$\boxed{\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}}$$

Here, independent → t, y

dependent → u

Parameter → ν

Now, our solution should not dependent on t, y. Thus, we non-dimensionalize the equation.

Length scale = No length scale

Velocity scale = U

Time scale = No

} Expect similarity solution.

We cannot non-dimensionalize length & time! So we combine independent variables and parameters to make new ~~non-dimensi~~ reference scales.

(non dimensional independent variable, η)

$$\frac{u}{U} = f(\eta) = g(t, y, \nu)$$

1 independent variable

2 independent variable

Here, $\eta = g(y, t, \nu)$.

To find the relation, assume $\eta = \nu^{\alpha} y^{\beta} t^{\gamma}$

Define

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial t}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) = \frac{\partial^2}{\partial y \partial \eta} \frac{\partial}{\partial y}\left(\frac{\partial}{\partial \eta}\right) \frac{d\eta}{dy} + \frac{\partial}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2}$$

$$= \left[\frac{\partial}{\partial \eta} \frac{\partial}{\partial \eta} \cdot \frac{d\eta}{dy}\right] \cdot \frac{d\eta}{dy} + \frac{\partial}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2}$$

Now, start finding relation of η by first considering y, t alone

$$\eta = c y t^{m} \qquad c = c(\nu)$$

$$\frac{\partial \eta}{\partial t} = m c y t^{m-1} = m \eta t^{-1}$$

$$\frac{\partial \eta}{\partial y} \frac{d\eta}{dy} = c t^{m}$$

$$\frac{\partial^2 \eta}{\partial y^2} = 0$$

Consider the original solution equation. $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$.

LHS $\Rightarrow$ $\frac{\partial u}{\partial t} = U \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial t} = U f'(m \eta t^{-1})$

$= U \left[f''(c^2 t^{m})^2 + 0\right] = U f'' c^2 t^{2m}$

RHS $\Rightarrow$ $\frac{\partial^2 u}{\partial y^2} = U \left[\frac{\partial^2 f}{\partial \eta^2} \cdot \left(\frac{d\eta}{dy}\right)^2\right]$

Equating,

$$U f' m \eta t^{-1} = \nu U f'' c^2 t^{2m}$$

$$f'' = m \eta t^{-1}$$

$$f'' - \frac{m \eta t^{-1}}{\nu c^2 t^{2m}} \cdot f' = 0 \qquad f' = \frac{\partial f}{\partial \eta}$$

* Equation should be independent of t, ν (dimensional variable).

To make the equation dimensionless,

$$2m = -1 \Rightarrow \boxed{m = -1/2}.$$

Now, $$f'' - \underbrace{\frac{1}{2}\frac{\eta}{\sqrt{c^2}}}\cdot f' = 0$$

Again, the coefficient should be constant (no function of variables)

$$\therefore c^2 \propto \frac{1}{\nu} \Rightarrow c^2 = \frac{1}{4\nu}$$

* any constant

$$\boxed{c^2 = \frac{1}{2\sqrt{\nu}}}$$

Thus, the solution becomes,

$$\boxed{f'' - 2\eta f' = 0} \quad (\leftarrow \text{ODE, Linear}) \qquad \boxed{\eta = \frac{y}{2\sqrt{\nu t}}}$$

Now, convert the boundary conditions for this new variable

$$u(0,y) = 0 \longrightarrow f(\infty) = 0 \quad \left(\eta = \frac{y}{2\sqrt{\nu 0}}\right)$$

$$u(t,0) = U \longrightarrow f(0) = 1 \quad \left(\eta = \frac{y}{U}\right)$$

$$u(t,\infty) = 0 \longrightarrow f(\infty) = 0$$

Here, only two B.C emerge because the number of independent variables is reduced

Solve.

Integrate the similar solution,

$$\int \frac{f''}{f'} d\eta = -\int 2\eta \, d\eta$$

$$\ln f' = -\eta^2 + \ln A$$

$$f' = Ae^{-\eta^2}$$

Error function.

$$\int_0^\infty e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$$

Integrate

$$f(\eta) = \int_0^{\eta} A e^{-\eta^2} d\eta + B$$

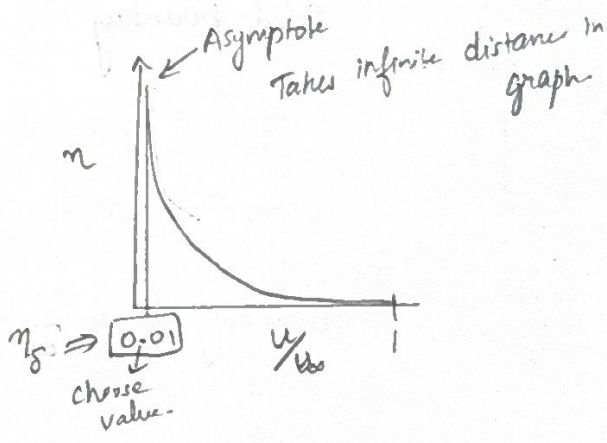
$\eta = 0, \quad f(0) = 1 \Rightarrow \boxed{B = 1}$

$\eta = \infty, \quad f(\infty) = 0 \Rightarrow \quad 0 = \int_0^{\infty} A e^{-\eta^2} d\eta + 1$

By standard integrals, $\boxed{A = -\frac{2}{\sqrt{\pi}}}$

$$\therefore \quad f(\eta) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\eta^2} d\eta$$

$$\boxed{f(\eta) = 1 - erf(\eta)}$$



Diffusive length (non-dimensional) (δ)

Impulse is given to plate → momentum is diffused into the fluid

It will never settle into a steady solution.

Thus, Choose the diffusive length to be the value where we are satisfied as having achieved end state

Now, $\eta_{0.01} = 1.82 = \frac{\delta}{2\sqrt{\nu t}}$

$$\therefore \quad \boxed{\delta = 3.64\sqrt{\nu t}}$$

Dimensional PDE ~~Similar~~ parent → Non-dimensional ODE → Solve → N.d solution → Substitution → Required solution

Reference scale

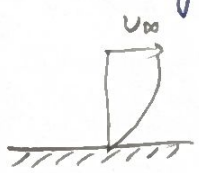
Thin shear layer

Shear – Velocity variation perpendicular to the mean flow.

We consider the viscosity as the reason for creation of thin shear layer

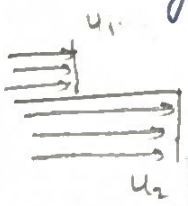
Viscosity ⟶ No slip ⟶ Shear

Solid boundary



Boundary layer

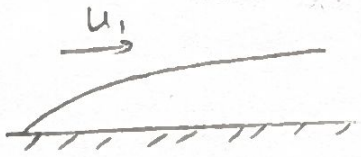
Free boundary

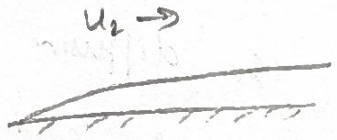


Free shear layer

If the velocity of free stream is very high, then the boundary layer or shear layer becomes thinner.

$u_2 > u_1$





Here, advection dominates and "takes" away transport material.

This is quantified by Reynolds number

In Bow shock, vorticity is induced primarily by the viscous effects.

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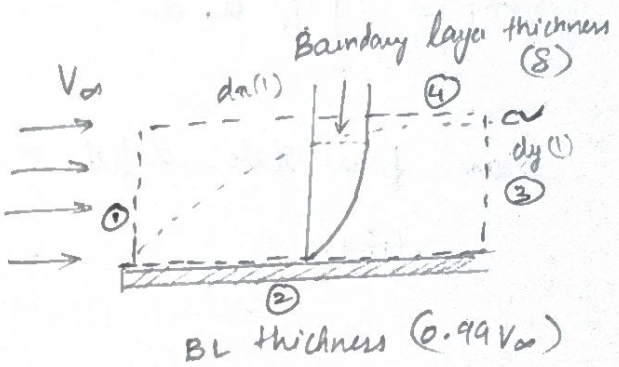
BOUNDARY LAYER

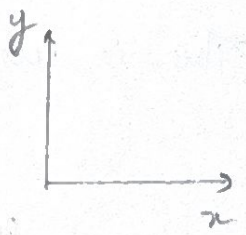
Control volume analysis of B.L.

Is there any mass flow across the boundary layer?



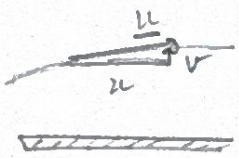
Assume.

* steady
* Mass conservation.



According to mass conservation equation, $\iint_{CS} \rho(\underline{u}\cdot\hat{n})\,dA = 0$

$$\iint_{CS_1} \rho(\underline{u}\cdot\hat{n})\,dA = -\iint_{CS_1} \rho_\infty V_\infty\, dy(1)$$



$$\iint_{CS_2} \rho(\underline{u}\cdot\hat{n})\,dA = 0 \qquad (\underline{u}\cdot\hat{n} = 0)$$

$$\iint_{CS_3} \rho(\underline{u}\cdot\hat{n})\,dA = \iint_{CS_3} \rho(y)\, u(y)\, dy$$

$$\iint_{CS_4} \rho(\underline{u}\cdot\hat{n})\,dA = \iint_{CS_4} \rho\, v(x)\, dx \qquad (v \text{ is the y-component of velocity})$$

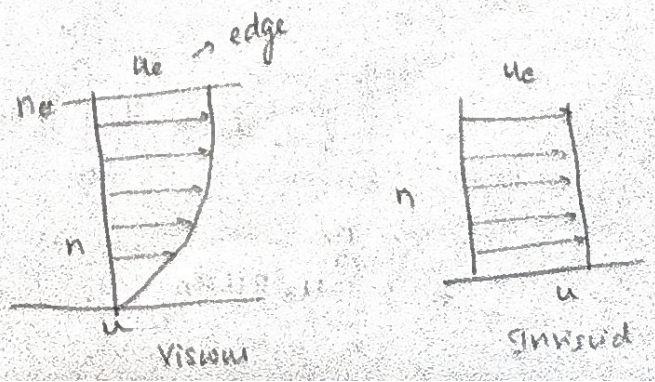
Combining these equations,

~~$\iint \rho(\underline{u}\cdot\hat{n})\,dA$~~ = $\int \rho\, v(x)\,dx = \int [\rho_\infty u_\infty - \rho(y)\, u(y)]\,dy$

Thus, $v(x) \neq 0$

Integral properties of BL.

1) Mass flow rate comparison



$$\dot{m}_v = \int_0^{n_e} \rho(n)\, u(n)\, dn$$

$$\dot{m}_i = \int_0^{n_e} \rho_e\, u_e\, dn$$

Mass flow rate defect $= \dot{m}_i - \dot{m}_v$

$$(\dot{m}_d) = \int_0^{n_e} (\rho_e u_e - \rho u)\, dn$$

This is used to define a new type of thickness. δ^* (Displacement thickness)

$$\rho_e u_e \delta^* = \int_0^{n_e} (\rho_e u_e - \rho u)\, dn$$

$$\boxed{\delta^* = \int \left(1 - \frac{\rho u}{\rho_e u_e}\right) dn}$$

δ^* gives the area that is excess in inviscid flow comparing to the viscous flow.

2) Momentum flow rate (Force) comparison

$$F_v = \int_0^{n_e} \rho u^2 \, dy\, dn$$

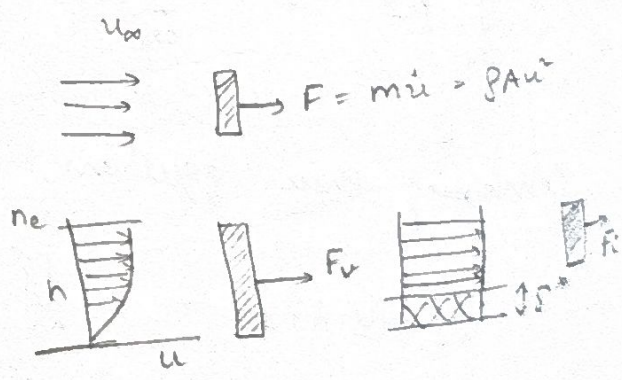
$$F_i = \int_{\delta^*}^{n_e} \rho_e u_e^2 \, dn = \int_0^{n_e} \rho_e u_e^2 \, dn - \int_0^{\delta^*} \rho_e u_e^2 \, dn$$

$$= \int_0^{n_e} \rho_e u_e^2 \, dn - u_e \underbrace{\int_0^{\delta^*} \rho_e u_e \, dn}_{\text{cont.}}$$

$$= \int_0^{n_e} \rho_e u_e^2 \, dn - u_e (\rho_e u_e \delta^*)$$

$$= \int_0^{n_e} \rho_e u_e^2 \, dn - u_e \int_0^{n_e} (\rho_e u_e - \rho u)\, dn$$

$$= \int_0^{n_e} u_e \rho u \, dn$$



$$\mathbb{P} = F_i - F_v = \int_0^{n_e} u_e \rho u \, dn - \int_0^{n_e} \rho u^2 \, dn$$

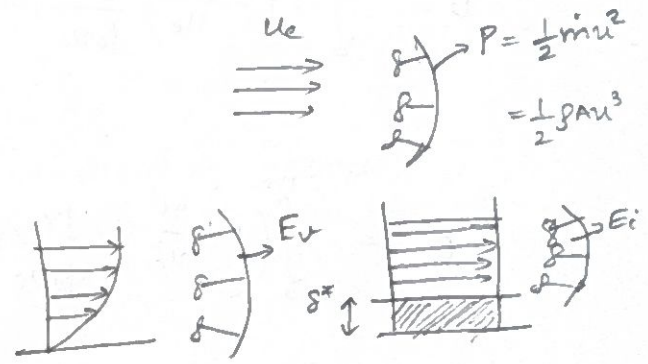
Momentum deficit

$$\boxed{\mathbb{P} = \int_0^{n_e} (u_e - u) \rho u \, dn}$$

* This is attributed to drag force.

3) Energy flow rate (Power)

* For a same mass flow rate, compare the energy difference



$$\dot{E}_v = \frac{1}{2}\int_0^{n_e} \rho u^3 \, dn$$

$$\dot{E}_i = \frac{1}{2}\int_{\delta^*}^{n_e} \rho_e u_e^3 \, dn = \frac{1}{2}\int_0^{n_e} \rho_e u_e^3 \, dn - \frac{1}{2}\int_0^{\delta^*} \rho_e u_e^3 \, dn$$

$$= \frac{1}{2}\int_0^{n_e} \rho_e u_e^3 \, dn - \frac{1}{2}u_e^2 \int_0^{\delta^*} \rho_e u_e \, dn$$

$$= \frac{1}{2}\int_0^{n_e} \rho_e u_e^3 \, dn - \frac{1}{2}u_e^2 (\rho_e u_e \delta^*)$$

$$= \frac{1}{2}\int_0^{n_e} \rho_e u_e^3 \, dn - \frac{1}{2}u_e^2 \int_0^{n_e} (\rho_e u_e - \rho u) \, dn$$

$$= \frac{1}{2}\int_0^{n_e} u_e^2 \rho u \, dn$$

Kinetic energy deficit

$$\mathbb{K} = -\frac{1}{2}\int_0^{n_e} \rho_e u_e^3 \, dn + \frac{1}{2}\int_0^{n_e} u_e^2 \rho u \, dn$$

$$\boxed{\mathbb{K} = \frac{1}{2}\int_0^{n_e} (u_e^2 - u^2) \rho u \, dn}$$

Integral thickness

δ^* – displacement thickness (measure of the distance of the shear layer centre from the wall)

θ – Momentum thickness (measure of thickness of the shear layer)

θ^* – Energy thickness.

$$\theta = \int_0^{n_e} \left(1 - \frac{u}{u_e}\right) \frac{\rho u}{\rho_e u_e} \, dn$$

$$\theta^* = \int_0^{n_e} \left(1 - \frac{u^2}{u_e^2}\right) \frac{\rho u}{\rho_e u_e} \, dn$$

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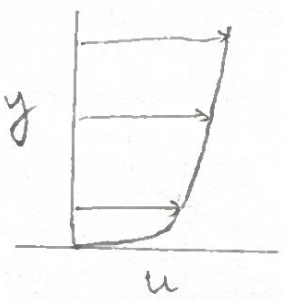
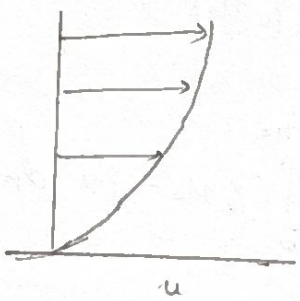
Shape factor, $H = \frac{\delta^*}{\theta}$ → compares the displacement & momentum thickness

* depends only on the shape of shear profile.



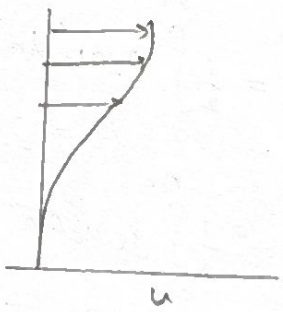
Turbulent flow

$H = 1.4$



Laminar flow

$H = 2.6$



Starting of flow seperation

$H \approx 4.0$

How to model the viscous flow

At $Re \gg 1$ ⇒ Thin shear layer

$Re \longrightarrow \infty$, $\mu \ll 1$

Non-dimensional $\nabla' u' = 0$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

$$x' = \frac{x}{L}, \quad y' = \frac{y}{L}, \quad u' = \frac{u}{U_\infty}, \quad v' = \frac{v}{U_\infty}$$

$$p' = \frac{P - P_\infty}{\frac{1}{2} \rho_\infty U_\infty^2}$$

Global scale

$$\rightarrow \quad \frac{Du'}{Dt} = -\nabla' p' + \frac{1}{Re}(\nabla')^2 u' \qquad (\nabla')^2 u' \to 0$$

Thus, for $Re \to \infty$, the flow is modelled as inviscid flow.

But, it fails to predict drag (De-Alembeit's Paradox)

Prandtl, 1904

- Introduce boundary layer concept
- Even as $Re \to \infty$, B.L can't be neglected.

Divide the total flow field around airfoil = inviscid + viscous.
outer flow inner flow

Inside B.L

$$\frac{\partial}{\partial y} >> \frac{\partial}{\partial x} \quad ; \quad \frac{\delta}{L} << 1$$

Velocity has to go from zero to U_∞ in very short y distance

→ Here, viscous forces are of same order of inertia force.

$$\nu \frac{\partial^2 u}{\partial y^2} \sim u \frac{\partial u}{\partial x}$$

$$\nu \frac{U_\infty}{\delta^2} \sim \frac{U_\infty^2}{L}$$

$$\boxed{\frac{\delta}{L} \propto \frac{1}{\sqrt{Re}}}$$

Using continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{U_\infty}{L} \sim \frac{v}{\delta}$$

$$\frac{v}{U_\infty} \propto \frac{\delta}{L} \quad \Rightarrow \quad \boxed{\frac{v}{U_\infty} \propto \frac{1}{\sqrt{Re}}}$$

The above two are different from global scale

Thus, local scaling is done,

$$u^* = \frac{u}{U_\infty}, \quad x^* = \frac{x}{L}.$$

$$\Rightarrow \quad v^* = \frac{v}{U_\infty}\sqrt{Re} \qquad y^* = \frac{y}{\delta} = \frac{y}{L}\sqrt{Re}$$

Using these to non-dimensionalize,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

1. $$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

2. $$\frac{1}{Re}\left[u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*}\right] = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{1}{Re}\frac{\partial^2 v^*}{\partial y^{*2}}$$

Now, as $Re \to \infty$, we end up with

a) $$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial y^{*2}}$$

b) $$0 = -\frac{\partial p^*}{\partial y^*}$$

Prandtl Boundary Layer eqn

(Valid for $Re \to \infty$)

Conservation equations.

c) $$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*}$

Implications

1) $$\frac{\partial p^*}{\partial y^*} \approx 0$$

2) Parabolic equation.

3) Non-dimensional boundary layer equation is independent of Re

Dimensional form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$ Approximately → Thus parallel streamline may not be the case inside BL.

Independent variable :- x, y

Dependent variable :- u, v

$P(x,y) \approx P(x)$ ← only possible due to geometry change in inviscid solution

Pressure is not influenced by the B.L.

Pressure is a parameter (input.) for Boundary layer calculation.

Boundary condition

based on derivative in eqn.

$u(x) \rightarrow 1$

$u(y) \rightarrow \infty$

$v(x) \rightarrow 0$

$v(y) \rightarrow 1$

To decide where to apply Boundary condition, nature of eqn. (parabolic, elliptic) is needed.

4

Problem

Prandtl proposed that global scale inside boundary layer does not work; and suggested local scale

$\delta << L$

$$\frac{\partial}{\partial y} >>> \frac{\partial}{\partial x}$$

$$\boxed{\frac{\delta}{L} \propto \frac{1}{\sqrt{Re}}}$$

$$\frac{v}{v_\infty} \propto \frac{1}{\sqrt{Re}}$$

For $Re \to \infty$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

PDE, non-linear

$$0 = \frac{\partial p}{\partial y}$$

PDE properties

$$A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x \partial y^2} + C\frac{\partial^2 f}{\partial y^2} + \dots = 0$$

$B^2 - 4AC < 0 \rightarrow$ elliptic

$= 0 \rightarrow$ Parabolic

$> 0 \rightarrow$ Hyperbolic

Steady Navier Stokes eqn $\Rightarrow$ Elliptic

Steady BL eqn $\Rightarrow$ Parabolic.

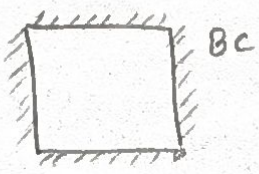
* Equilibrium — elliptic
* Propagation (marching) — Parabolic, Hyperbolic

Marching is only in x

Elliptic

→ BVP $\nabla^2 u = 0$

→ Closed domain $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

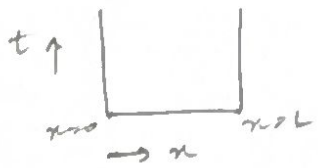


→ Every point in the domain is influenced by all other point.

4 BC needs to be mentioned.

Parabolic

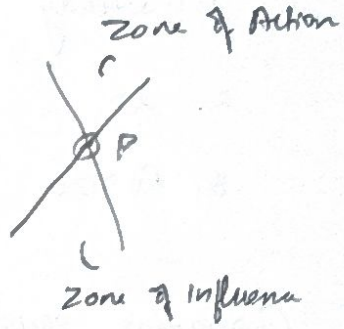
* → $\frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2}$



→ The solution is dependent on the solution at every point in the past.

→ IVP and BVP

→ Influence is one direction (No upstream influence).



Hyperbolic

* $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f^2}{\partial x^2}$

* IVP & BVP

→ In BL equation, there is no time variable.

→ But, x acts like time & variable.

↳ The solution at x_0 influences solution at x_1.

Again, Boundary condition,

		Non-dimensional
$u(y) \Rightarrow$	$u(x,0) = 0$	0
	$u(x,\infty) = U_\infty$	1
$v(y) \Rightarrow$	$v(x,0) = 0$	0
$u(x) \Rightarrow$	$u(x_{in}, y) = U_{in}$	$\frac{U_{in}}{U_\infty}$

Now, we want to solve Prandtl BL equation for a specific geometry

FLOW OVER A SEMI-INFINITE FLAT PLATE

when solving BL equation, continuity equation is also important

Assumptions

1. Steady
2. 2D
3. $Re \gg 0$

Governing equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{——— (1)}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P_e}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$ ($\partial P_e \to 0$)

$$0 = \frac{\partial P}{\partial y}$$

Here $\frac{\partial P_e}{\partial x} = 0$. But then how does flow take place?

→ Due to inertia of fluid flows.

Is it possible to get <u>self similar solution</u>? Check. (1 independent variable reduces.)

Velocity scale → U_∞ ✓

Length scale → NA

Time scale → NA

We use <u>stream function</u> ⇒ To enforce continuity equation automatically.

Define $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$

Eq① can be written as

$$\left(\frac{\partial \psi}{\partial y}\right)\frac{\partial^2 \psi}{\partial x \partial y} - \left(\frac{\partial \psi}{\partial x}\right)\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3} \qquad \text{——— (2)}$$

Order of eqn increased to accomodate continuity eqn

Similarity soln.

$$\frac{u}{U_\infty} = F(\eta) \qquad \eta = \frac{y}{\delta}$$

$$\boxed{\delta \propto \sqrt{\nu t}\,, \quad t = \frac{x}{U_\infty}}$$

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_\infty}}} = y\sqrt{\frac{U_\infty}{\nu x}} \Rightarrow d\eta = dy\sqrt{\frac{U_\infty}{\nu x}}$$

To find ψ

$$\psi = \int u\,dy = \int U_\infty F(\eta)\,dy$$

$$= \int U_\infty F(\eta)\frac{d\eta}{\sqrt{\frac{U_\infty}{\nu x}}}$$

$$\psi = \sqrt{U_\infty \nu x}\underbrace{\int F(\eta)d\eta}_{\longrightarrow f(\eta)}$$

Here, $f(\eta)$ is a non-dimensional stream function

$$\boxed{\psi = \sqrt{U_\infty \nu x}\, f(\eta)}$$

Here, $\psi(x,\eta)$ and $\eta(x,y)$.

We now try to satisfy eq ②-

Thus,

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial \eta}\frac{d\eta}{dx} + \frac{\partial \psi}{\partial x}\frac{dx}{dx}$$

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial \psi}{\partial x}\frac{dx}{dy}^{\,0}$$

$$\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial y}\right)\frac{d\eta}{dy} + \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)\frac{dx}{dy}^{\,0} \qquad \text{—— ③}$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial y^2}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial^2 \psi}{\partial y^2}\right)\frac{d\eta}{dy} + 0$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial \psi}{\partial y}\right)\frac{d\eta}{dx} + \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right)\frac{dx}{dx}.$$

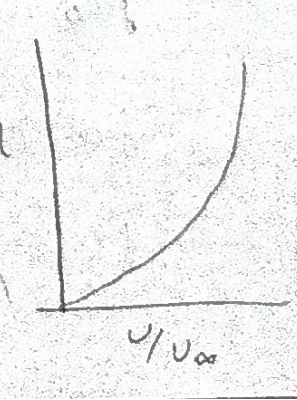
Substituting eq ③ into eq ② after algebra, we get

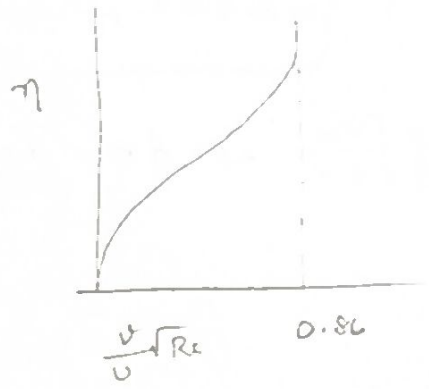
$$\boxed{f''' + \frac{1}{2} f f'' = 0}$$

ODE

Non-linear

$$f' = \frac{\partial f}{\partial \eta}$$





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Boundary layer (recap)

For $Re \gg 1$ $\quad \frac{\delta}{L} \ll 1$

$$\frac{\delta}{x} \propto \frac{1}{\sqrt{Re_x}} \qquad \frac{v}{U_\infty} \propto \frac{1}{\sqrt{Re_x}}$$

Prandtl's boundary layer equation is parabolic in space. Thus, it is solved using marching technique.

Initial condition is needed.

Self similar solution

→ Non-linear equations won't have closed form solution analytically

→ It often requires numerical integration.

→ Thus, self-similar solutions provide a closed form solution

BL over flat plate

$$f''' + \frac{1}{2} f f'' = 0$$

$f'(0) = 0$

$f'(\infty) = 1$

$f(0) = 0$

$$u = \frac{\partial \psi}{\partial y} = U_\infty f'$$

$$v = \frac{1}{2}\sqrt{\frac{U_\infty \nu}{x}}(\eta f' - f)$$

* Self-similar equations are independent of initial boundary condition.

* Self-similar equation are not parabolic.

Numerical solution of BL eq (Assignment)

$$f''' + \frac{1}{2} f f'' = 0$$

BC

$\eta = 0 \quad f(0) = 0, \quad f'(0) = 0$

$\eta \to \infty \quad f'(\infty) = 1$

- Third order, non-linear ODE
- Boundary value problem. (Difficult :- No solution, Unique soln, Multiple soln)

Procedure

* Convert the above into 3 linear ODE.

$$f' = g$$

$$g' = f'' = h$$

$$h' = f''' = -\frac{1}{2} f f'' = -\frac{1}{2} f h$$

* New BC. for $\eta = 0$

$$f(0) = 0$$

$$f'(0) = g(0) = 0$$

$\eta \to \infty$

$$f'(\infty) = g(\infty) = 1$$

* Shooting method.

Assume we convert BVP to Initial value problem.

$$f(0) = 0$$

$$g(0) = 0$$

$$h(0) = \text{Assume}$$

Solve IVP → check for convergence → refine assumption.

IVP. (discretization)

- Euler, RK

$$\frac{df}{d\eta} = g$$

$$\frac{dg}{d\eta} = h$$

From Taylor's series

$$f(i) = f(i-1) + \Delta\eta\, g^{(i-1)}$$

$$\frac{f(i) - f(i-1)}{\Delta\eta} = g \quad \checkmark$$

$i = N$, $\eta \to \infty$

$i = 0$, $\eta = 0$

$$g(i) = g(i-1) + \Delta\eta\, h^{(i-1)}$$

$$\frac{g(i) - g(i-1)}{\Delta\eta} = h^{(i-1)} \quad \checkmark$$

Check.

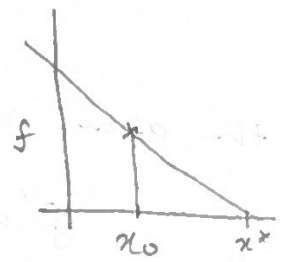
$g(N) - 1 = 0$ $\quad \varepsilon\ 10^{-5}$

$$h_i = h(i-1) + \Delta\eta\left[-\frac{1}{2} f(i-1)\, h(i-1)\right] \quad \checkmark$$

Newton - Raphson method (refine)

- Root finding method

$f(x_0) \neq 0$ $\quad x_0$ is not a solution



$$f(x_1) = f(x_0) + (x_1 - x_0)\, f'(x_0) \quad \text{(Taylor series)}$$

To get $f(x_1) = 0$,

$$-f(x_0) = (x_1 - x_0)\, f'(x_0)$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Now, $G(N) = g(N) - 1 = 0$

$$h_{(1)}^{k+1} = h_{(1)}^{k} - \frac{G^{k}(N)}{G'^{k}(N)}$$

$$G^{k}(N) = g^{k}(N) - 1$$

$$G'^{k}(N) = \frac{g^{k}(N) - g^{k-1}(N)}{h_{(1)}^{k} - h_{(1)}^{k-1}}$$

Here, to obtain h^{k+1}, h^k and h^{k-1} is needed.

Thus, first two values are assumed.

Run IVP ⟶ Newton Raphson

⟵

$\Delta h \approx 10^{-6}(\varepsilon)$.

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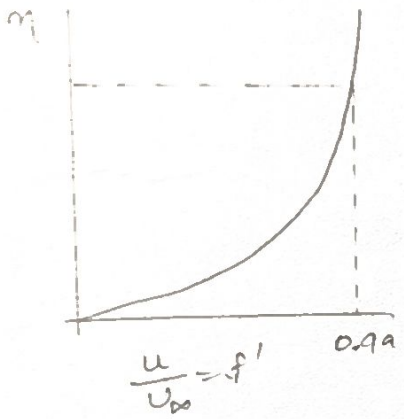
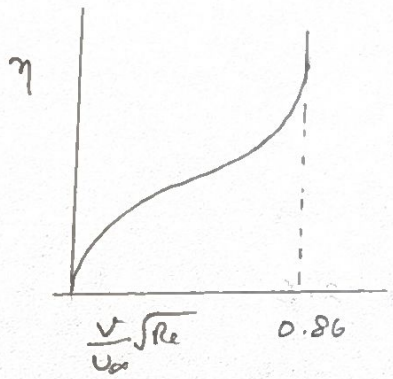
Solution to BL over flat plate





BL thickness (99% u_∞)

$$\eta_{0.99} = \frac{y}{\delta} = y\sqrt{\frac{u_\infty}{\nu x}} \approx 5 \quad \Rightarrow \quad d\eta = \frac{dy}{\sqrt{\frac{\nu x}{Re\, u_\infty}}}$$

$$y_{bl} = \delta = 5\sqrt{\frac{\nu x}{Re\, u_\infty}}$$

$$\boxed{\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}}$$

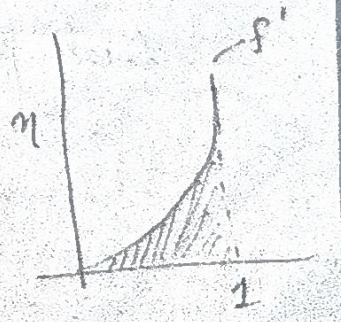
Displacement thickness (δ^*)

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy$$

$$\delta^* = \int_0^\infty [1 - f'(\eta)] \sqrt{\frac{\nu x}{u_\infty}}\, d\eta$$

$$\delta^* = \sqrt{\frac{\nu x}{u_\infty}} \left[\int_0^\infty [1 - f'(\eta)]\, d\eta\right]$$

To be obtained by numerical integration



$$\frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}}$$

Momentum thickness (θ)

$$\theta = \int_0^\infty \left(1 - \frac{u}{U_\infty}\right) \frac{\rho u}{\rho U_\infty} dy = \left[\int_0^\infty (1 - f') f' \, d\eta\right] \sqrt{\frac{\nu x}{U_\infty}}$$

0.664

$$\frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

Shape factor (H) $= \frac{\delta^*}{\theta} = 2.6$

Local skin friction coefficient $\}$ $\tau_w = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^0$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\partial u}{\partial \eta} \frac{d\eta}{dy} = (U_\infty f'') \sqrt{\frac{U_\infty}{\nu x}}$$

$$\therefore \quad \tau_w = \frac{\mu U_\infty f''(0)}{\sqrt{\frac{\nu x}{U_\infty}}} = \frac{\mu U_\infty^{3/2} f''(0)}{\sqrt{\nu x}}$$

$$\tau_w = \frac{0.332 \rho U_\infty^2}{\sqrt{Re_x}}$$

$\tau_w \propto U_\infty^{3/2}$

$\tau_w \propto \frac{1}{\sqrt{x}}$

Local friction coefficient, $C_f = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2}$

$$C_f = \frac{0.332}{\sqrt{Re_x}}$$

Drag force per unit width at one side of plate & length L.

$$D' = \int_0^L \tau_w \, dx$$

$$\boxed{D' = \frac{0.664 \, \rho U_\infty^2 L}{\sqrt{Re_L}}}$$

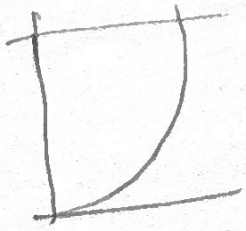
(τ_w goes to infinity when $x=0$. But not so when integrated → called as removable singularity)

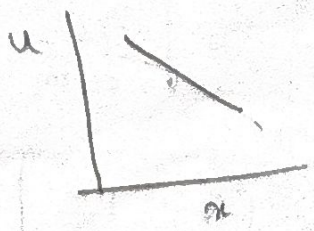
$$C_d = \frac{D'}{\frac{1}{2}\rho U_\infty^2 L} = \frac{1.332}{\sqrt{Re_L}}$$

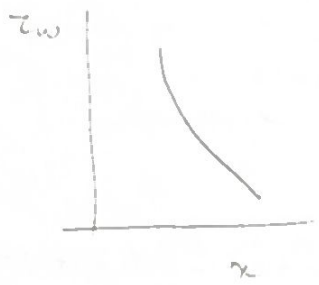
Summary

quantity	Length	Velocity	Viscosity
δ, δ^*, θ $\left(\propto \frac{x}{\sqrt{Re_x}}\right)$	$\sqrt{x}$	$\frac{1}{\sqrt{U_\infty}}$	$\sqrt{\nu}$
$\tau_w \propto \frac{\rho U_\infty^2}{\sqrt{Re_x}}$	$\frac{1}{\sqrt{x}}$	$U_\infty^{3/2}$	$\sqrt{\nu}$
$C_f \propto \frac{1}{\sqrt{Re_x}}$	$\frac{1}{\sqrt{x}}$	$\frac{1}{\sqrt{U_\infty}}$	$\sqrt{\nu}$
$D' \propto \frac{\rho U_\infty^2 L}{\sqrt{Re_L}}$	$\sqrt{L}$	$U_\infty^{3/2}$	$\sqrt{\nu}$
$C_d \propto \frac{1}{\sqrt{Re_L}}$	$\frac{1}{\sqrt{L}}$	$\frac{1}{\sqrt{U_\infty}}$	$\sqrt{\nu}$

Interpretation







Drag comparison,

$Re \ll 1$

$Re \gg 1$

$D' \propto u$ (friction drag)

$D' \propto u^2$ (Pressure drag)

$D' \propto \frac{1}{u^2}$ (Induced drag)

$D' \propto U^{3/2}$ (Laminar incompressible)

Incompressible, $c_f = f(Re)$

Compressible, $c_f = f(Re, M, Pr, T)$

※ Can we have self similar solution for BL when there is pressure gradient in outside flow?

Answer

Falkner - Skan flow

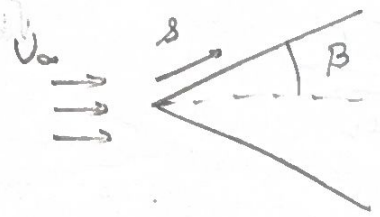
BL is a local phenomena

$U_e(s) = cs^m$ (Power law) ← edge velocity

$s \rightarrow$ streamwise direction

What physical configuration will give this kind of velocity? → wedge flow

$$U_e(s) = cs^{\left(\frac{\beta}{180-\beta}\right)}, \quad m = \frac{\beta}{180-\beta}$$



Power law velocity / Wedge flow

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Falkner - Skan boundary layer thickness,

$$\delta_{FS} = \sqrt{\frac{\nu s}{u_e}} = \sqrt{\frac{\nu s}{cs^m}} = \sqrt{\frac{\nu}{c}}\, s^{\frac{1-m}{2}}$$

$$\boxed{\delta_{FS} = \sqrt{\frac{\nu}{c}}\, s^{\frac{1-m}{2}}}$$

Now, $\frac{-dp}{ds} = u_e \frac{du_e}{ds} \Rightarrow \frac{-dp}{ds} = cs^m mcs^{m-1}$

$$\boxed{\frac{-dp}{ds} = mc^2 s^{2m-1}}$$

Wedge Flow (Potential flow)

Assignment ← Solve numerically

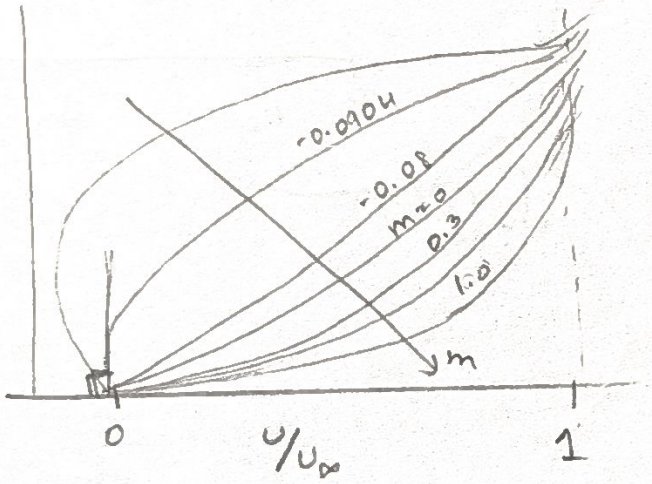
$$f''' + \left(\frac{m+1}{2}\right) f f'' + m\left[1-(f')^2\right] = 0 \qquad \text{(Self similar solution)}$$

BC.

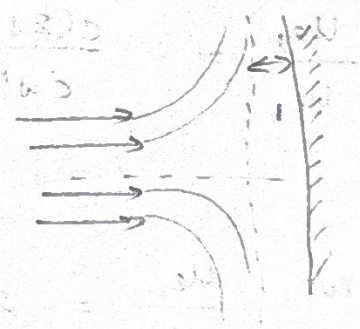
@ $\eta = 0$ $\quad f(0) = f'(0) = 0$

$\eta = 1$ $\quad f'(\infty) = 1$

Solution.



Case I: $\beta = 90^\circ$ $(m=1)$



$\delta_{Fs} = C$

$-\frac{dp}{ds} = c^2 s > 0$ (Favourable pressure gradient)

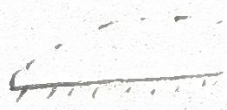
* velocity profile, $u = cs$
  velocity keeps increasing.

* At stagnation point, boundary layer thickness is finite.

* This solution is helpful for calculation near stagnation point.

Case II: $\beta = 0^\circ$ $(m=0)$

$\delta_{Fs} = \sqrt{\frac{\nu s}{c}}$



$-\frac{dp}{ds} = 0$

=> Favorable pressure gradient suppresses boundary layer growth.

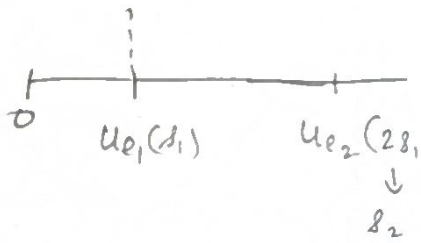
Case III: $\beta = -17.82^\circ$ $(m = -0.0904)$

* incipient of flow seperation
(starting of flow seperation).

$-\frac{dp}{ds} = < 0$ (Adverse pressure gradient)

What is the adverse pressure gradient value ~~at~~ with laminar flow can keep going without flow seperation.

$$\frac{u_{e_2}}{u_{e_1}} = \frac{c(2s)^m}{c s^m} = 2^m$$



Here, $\frac{u_{e_2}}{u_{e_1}} = 2^{-0.0904} = 0.9392$

$\simeq 94\%$.

Thus laminar flow velocity decrease by 6% over distance s_1. Any more and laminar flow will seperate.

For designing airfoil, this solution cannot be used directly (not self similar).

Thus, how to design airfoil?

Non self similar solution

Integral method

Interest → τ_w, D, δ^*

Integral Boundary Layer eqn.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \text{— ①}$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \qquad \text{— ②}$$

Now, $(u_e - u) \times$ eq ① − eq ②

$$\Rightarrow \frac{\partial[(u_e-u)\rho u]}{\partial x} + \frac{\partial}{\partial y}[(u_e-u)\rho v] = -(\rho_e u_e - \rho u)\frac{du_e}{dx} - \frac{\partial \tau}{\partial y}$$

Integrate the eqn. from 0 to y $(y > \delta)$.

$$\Rightarrow \frac{\partial}{\partial x}\int_0^y (u_e - u)\rho u\, dy + \cancel{[(u_e-u)\rho v]_0^y} = -\frac{du_e}{dx}\int_0^y (\rho_e u_e - \rho u)dy - [\tau]_0^y$$

$$\Rightarrow \frac{\partial}{\partial x}(\rho_e u_e^2 v) + 0 = -\dot{m}_d \frac{du_e}{dx} + \tau_w$$

$$\boxed{\frac{dP}{dx} = -\dot{m}_d \frac{du_e}{dx} + \tau_w}$$

[Integral Boundary Layer eqn]

Von Karman

If states the momentum deficite change as we move along x.

$$\frac{dP}{dx} = -\rho_e u_e \delta^* \frac{du_e}{dx} + \tau_w$$

$$= \delta^* \frac{dp}{dx} + \tau_w$$

The momentum deficit of an airfoil can be found by integrating from leading edge to the end of wake. And,

$$\Pi \propto D'$$

$$\frac{d}{dx}(\rho_e u_e^2 \theta) = -\rho_e u_e \delta^* \frac{du_e}{dx} + \tau_w$$

$$\rho_e u_e^2 \frac{d\theta}{dx} + \rho_e \theta 2u_e \frac{du_e}{dx} + u_e^2 \theta \frac{d\rho_e}{dx} = -\rho_e u_e \delta^* \frac{du_e}{dx} + \tau_w$$

From Euler equation,

$$\rho_e u_e \frac{du_e}{dx} = -\frac{dp}{dx} = \left(\frac{dp}{d\rho}\right)_s \frac{d\rho}{dx} = -a^2 \frac{d\rho}{dx}$$

$(\div \rho_e u_e^2)$

$$\Rightarrow \boxed{\frac{d\theta}{dx} + \frac{\theta}{u_e}\frac{du_e}{dx}\left[H + 2 - M_e^2\right] = \frac{C_f}{2}}$$

Mach number

Non-dimensional version

$$H = \frac{\delta^*}{\theta} \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$

shape factor

Class.

Non Self-Similar Boundary Layer (solution)

Eq in θ

(IVP)

Input $u_e(x)$, $u_{initial}$, $u_{e(x_{initial})}$, $\boxed{H \;\&\; C_f}$ (Assume incompressible) $M_e \approx 0$

Solution method.

1) Solve actual BL eq (non-integral form)

=> solve numerically

(PDE) Input : $u_e(s)$, $u(s_0)$, $v(s_0)$

Output : $u(s)$, $v(s)$ $\Longrightarrow$ $\delta^*, \theta, H, C_f, D$

2) Integral momentum eq (ODE)

- Numerical
- Analytical

Input : u_e, H, C_f

Output :

~ Numerical

Taylor's series

$$\theta_{i+1} = \theta_i + \left(\frac{d\theta}{ds}\right)_i \Delta s \quad \Longleftarrow \text{Discretization}$$

$\Delta u_e = u_{e,i+1} - u_{e,i}$

$\Delta s = s_{i+1} - s_i$

$$= \theta_i + \left[\frac{C_f}{2} - (H_i + 2)\frac{\theta_i}{u_{ei}}\frac{\Delta u_e}{\Delta s}\right]$$

n Thwaits Method laminal BL (numerical)

Idea

* ~~C_f, H from~~ For θ_i $\xrightarrow[\text{need}]{\text{we}}$ H & C_f from Falkner - Skan profile.

Now if we know m $\xrightarrow[\text{can}]{\text{we}}$ $\frac{\delta^*}{x}\sqrt{Re}$, $\frac{\theta}{x}\sqrt{Re}$, $\frac{C_f}{2}\sqrt{Re}$, H

How to find m?

$u_e(s) = Cs^m \Rightarrow \frac{du_e}{ds} = mCs^{m-1} = \frac{m}{s}u_e$

$\Rightarrow m = \frac{s}{u_e}\left(\frac{du_e}{ds}\right)$ ← Input (we dont know s. s depends on where origin is)

X

Thus,

$$\left[\frac{s}{u_e}\left(\frac{du_e}{ds}\right)\right]\left[\theta\sqrt{\frac{u_e}{\nu s}}\right]^2 = \boxed{\frac{\theta^2}{\nu}\frac{du_e}{ds} = \lambda}$$

examining θ & dependence (coordinate dependence)

Thus, for given m, we have

$$\frac{\delta^*}{x}\sqrt{Re}, \quad \frac{\theta}{x}\sqrt{Re}, \quad \frac{C_f}{2}\sqrt{Re}, \quad H, \quad \lambda$$

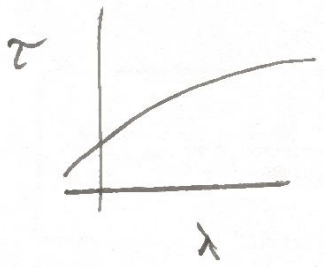
→ unique for given m

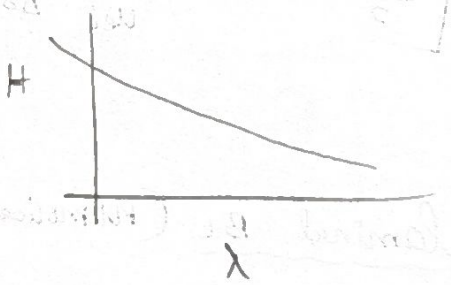
$\lambda \rightarrow$ non-dimensional pressure gradient

Also,

$$\left[\theta\sqrt{\frac{u_e}{\nu s}}\right]\left[\frac{C_f}{2}\sqrt{\frac{u_e s}{\nu}}\right] = \frac{u_e\theta}{\nu}\frac{C_f}{2} = \boxed{Re_\theta\frac{C_f}{2} = \tau}$$

For given location $\xrightarrow[\text{know}]{\text{we}}$ λ & $\lambda \xrightarrow[\text{know}]{\text{we}}$ H, τ





Once we know $\lambda \Rightarrow \tilde{H}, \tilde{\tau}$ → Approximates for Thwait's method

↑ (Non self similar flow)

$$\frac{d\theta}{ds} = \frac{C_f}{2} - (H+2)\frac{\theta}{u_e}\frac{du_e}{ds}$$

$$\Rightarrow \boxed{\frac{d\theta}{ds} = \frac{1}{Re_\theta}\tilde{\tau}(\lambda) - \left[\tilde{H}(\lambda)+2\right]\frac{\lambda}{Re_\theta}}$$

Solving numerically,

Input

θ_{in}

~ Simplified / Analytical Thwaits method.

$$\frac{d\theta}{ds} = \frac{1}{Re_\theta}\,\tilde{\tau}(\lambda) - \left[\tilde{H}(\lambda)+2\right]\frac{\lambda}{Re_\theta}$$

$$\frac{u_e\theta}{\nu}\frac{d\theta}{ds} = \left[\tilde{\tau}(\lambda) - \left[\tilde{H}(\lambda)+2\right]\lambda\right]$$

$$\frac{u_e}{\nu}\frac{d\theta^2}{ds} = 2\left[\tilde{\tau}(\lambda) - \left[\tilde{H}(\lambda)+2\right]\lambda\right]$$

$$\frac{u_e}{\nu}\frac{d\theta^2}{ds} = F_\theta(\lambda)$$

$$\boxed{u_e\,\frac{d(\theta^2/\nu)}{ds} = 0.45 - 6\lambda}$$

Simplification.

Now,

$$u_e\,\frac{d(\theta^2/\nu)}{ds} = 0.45 - 6\left(\frac{\theta^2}{\nu}\frac{du_e}{ds}\right)$$

$$\boxed{\frac{d}{ds}\left(\frac{\theta^2}{\nu}\right) + \frac{6}{u_e}\frac{du_e}{ds}\left(\frac{\theta^2}{\nu}\right) - \frac{0.45}{u_e} = 0}$$

Integrating factor, $e^{\int P\,dx} = e^{\int \frac{6}{u_e}\frac{du_e}{ds}\,ds} = e^{6\ln u_e} = u_e^6$

$$\Rightarrow \boxed{\frac{d}{ds}\left(u_e^6\,\frac{\theta^2}{\nu}\right) = 0.45\,u_e^5}$$

Integrate. $$u_e^6 \frac{\theta^2}{\nu} = 0.45 \int_0^s u_e^5 ds + \left(u_e^6 \frac{\theta^2}{\nu}\right)_{s=0}$$

Case I

if sharp edge body, at $s=0$, $\theta = 0$

Case II

if blunt nose body, $s=0$, $u_e = 0$ (stagnation point)

In both the cases, $\left(u_e^6 \frac{\theta^2}{\nu}\right)_{s=0} = 0$..

$$\therefore \quad \boxed{\frac{\theta^2}{\nu} = \frac{0.45}{u_e^6} \int_0^s u_e^5 ds} \quad \star$$

Procedure

1) Given $u_e(s)$

2) $u_e(s)$ Sol eq $\Rightarrow \theta$

3) For each s, find $\lambda = \frac{\theta^2}{\nu} \frac{du_e}{ds}$. $\frac{du_e}{ds} = 0$ for $u_e(s) = c$

4) From $\lambda \Rightarrow \tilde{H}$ & $\tilde{\tau}$ from empirical eqn.

5) $\tilde{\tau} \Rightarrow c_f$

$\tilde{H} \Rightarrow \delta^*$

Flat Plate

$$u_e(s) = c$$

$$\frac{\theta^2}{\nu} = \frac{0.45}{u_e^6}\int_0^s u_e^5 ds = \frac{0.45}{u_e} s + \vec{\phi}^{\,0} \quad \text{because flat plate}$$

$$\frac{\theta^2}{\nu} u_e = 0.45\, s \implies \boxed{\frac{\theta}{s} = \frac{\sqrt{0.45}}{\sqrt{Re}} = \frac{0.671}{\sqrt{Re}}} \quad \begin{pmatrix}\text{Blasius.}\\ 0.644.\end{pmatrix}$$

$$\lambda = 0 \implies \tilde{H}(0) = 2.61 \quad \& \quad \tilde{\tau}(0) = 0.22$$

$$\tilde{H}(\theta) = \frac{\delta^*}{\theta} = \frac{\delta^*}{s} \times \frac{s}{\theta} \implies \boxed{\frac{\delta^*}{s} = \frac{1.75}{\sqrt{Re}}} \quad \begin{pmatrix}\text{Blasius.}\\ 1.72\end{pmatrix}$$

$$\tilde{\tau}(\theta) = Re_\theta \frac{C_f}{2} = 0.22 \implies Re_s \frac{\theta}{s} \frac{C_f}{2} = 0.22$$

$$C_f = \frac{0.6571}{\sqrt{Re_s}} \quad \begin{pmatrix}\text{Blasius}\\ 0.664\end{pmatrix}$$

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Effect of pressure gradient and shear gradient on BL velocity profile

Eq in s direction,

$$\rho u \frac{\partial u}{\partial s} = -\frac{dP}{ds} + \frac{\partial \tau}{\partial n} = \rho_e u_e \frac{du_e}{ds} + \frac{\partial \tau}{\partial n}$$

$$\frac{\partial u}{\partial s} = \frac{\rho_e u_e}{\rho u}\frac{du_e}{ds} + \frac{1}{\rho u}\frac{\partial \tau}{\partial n}$$

Integrating from 0 to s_i

$$\boxed{\Delta u = \frac{\rho_e u_e}{\rho u}\Delta u_e + \frac{1}{\rho u}\frac{\partial \tau}{\partial n}\Delta s}$$

Pressure gradient — shear gradient

Effect of pressure gradient

1) Favourable $\frac{dp}{dx} < 0$

From Navier Stokes,

$$u\left(\frac{du}{ds}\right) = -\frac{dp}{ds}$$

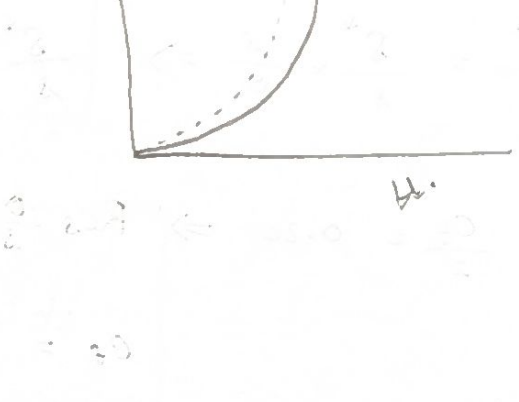
↙ acceleration is dependent on velocity.

$$\Delta u = \left(\frac{\rho_e u_e}{\rho u}\right) \Delta u_e$$

≥ 1

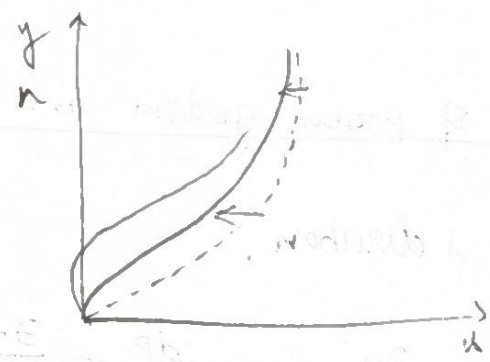
* Shear stress increases.

* Boundary layer growth decreases with distance



2) Adverse $\frac{dp}{ds} > 0$

* BL growth increases with distance



Effect of shear gradient

→ Shear stress is not constant along y in BL.

→ Shear gradient opposes BL growth (tries to suppress flow separation) Negative feedback.

⇒ Thus, boundary layer will not separate in adverse pressure gradient.

x-momen

$$u\frac{\partial u}{\partial s} + v\frac{\partial v}{\partial n} = -\frac{1}{\rho}\frac{dp}{ds} + \nu\left(\frac{\partial^2 u}{\partial n^2}\right)$$

at wall, $u = 0, v = 0$

$$\frac{1}{\rho}\frac{dp}{ds} = \nu \frac{\partial^2 u}{\partial n^2}$$

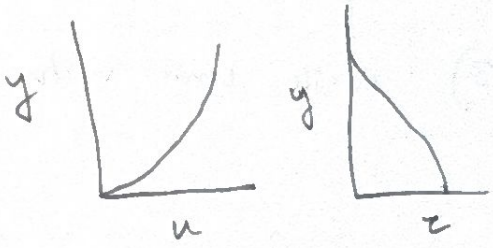
→ Curvature

$$\boxed{\frac{dp}{ds} = \mu\left(\frac{\partial^2 u}{\partial n^2}\right)}$$

Cases. What is curvature at edge of BL.

(i)

$$\left(\frac{\partial^2 u}{\partial n^2}\right)_\delta < 0$$



Inflection point $\frac{\partial^2 u}{\partial n^2} = 0.$

cases

(i) Flat plate

$$\frac{dp}{ds} = 0 \qquad \left(\frac{\partial^2 u}{\partial n^2}\right)_{n=0} = 0$$

(ii) $\frac{dp}{ds} < 0 \qquad \left(\frac{\partial^2 u}{\partial n^2}\right)_{n=0} < 0$ (Favorable pr. gradient)

(iii) $\frac{dp}{ds} > 0$

$\left(\frac{\partial^2 u}{\partial n^2}\right)_{n=0} > 0 \qquad \left(\frac{\partial^2 u}{\partial n^2}\right)_{n=\delta} < 0$ (Adverse pr. gradient)

Some where inflexion point lies. This leads to inviscid instability.

Mid term discussion

1a) $\nabla \cdot u = 0$ steady & unsteady

$\frac{D\rho}{Dt} = 0$

b) $\phi = \tau_{ij} \frac{\partial u_i}{\partial u_j}$

c)

d) $\nabla\rho \times \nabla P$

e) Total work done by surface forces

f) $\hat{e}_j \cdot$ (Momentum eqn)

2 a) Implicit superposition is not valid
highly sensitive to initial

b) irrotational.

3) Write unit vector also.

4) limits upto ∞.

5) Assignment

6)

b) $u^* = \frac{u}{U_\infty}\sqrt{Re}$ $\quad y^* = \frac{y}{L}\sqrt{Re}$

c)

BL	NS
parabola	elliptic.

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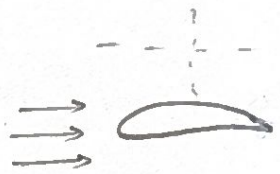
Flow seperation

When the flow is not following the surface.

Necessary condition for flow seperation.

1. Viscous flow
2. Adverse pressure gradient

Case I: Only adverse pressure gradient



In an airfoil in inviscid flow, the pressure decreases and velocity increases ~~in~~ near leading edge.

The pressure increases and velocity decreases in aft section.

But velocity cannot be negative.

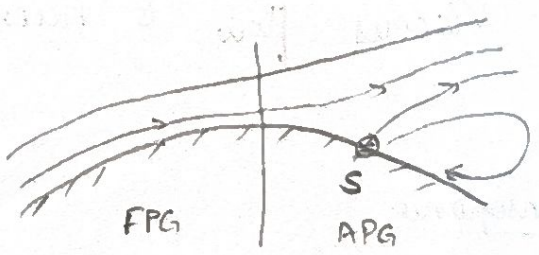
$$P + \frac{1}{2}\rho v^2 = P_0$$

(mv)

Case II: Only viscous.

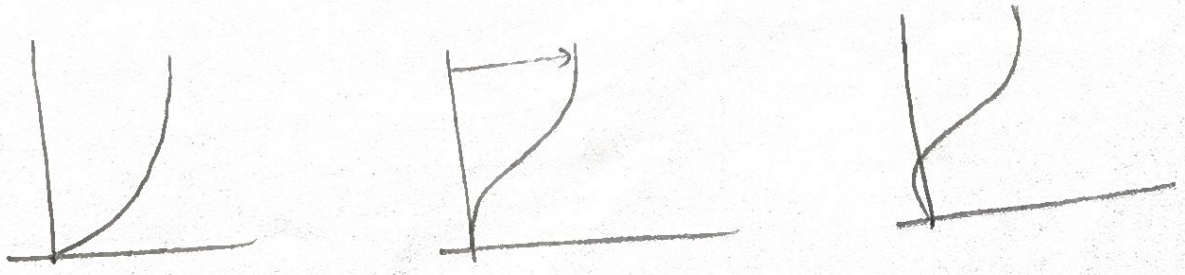
Here also, minimum velocity achievable is $V_{min} = 0$.

S → seperation point.



* Pressure in the seperated flow is approx. equal to P_s (Pressure at seperated point)

How to identify seperation point



Shear stress helps in identifying

$\tau > 0$ before separation

$\boxed{\tau = 0}$

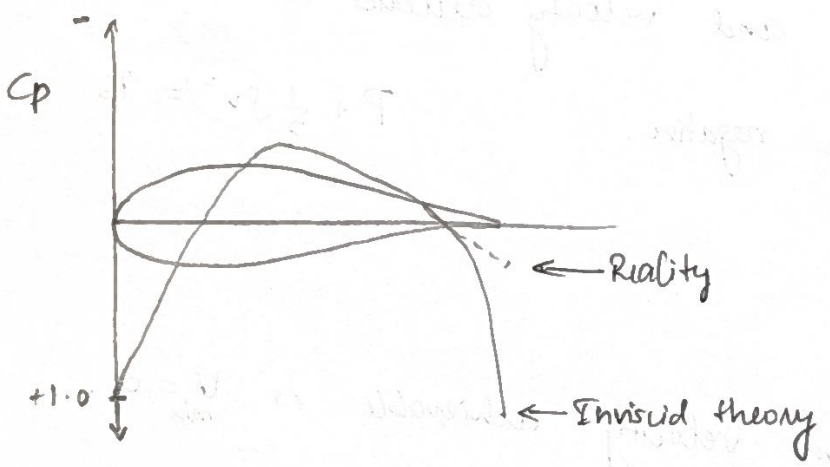
$\tau < 0$ after separation

Consequence of flow separation

Viscous flow
- Form drag (Pressure drag)
- Skin friction drag

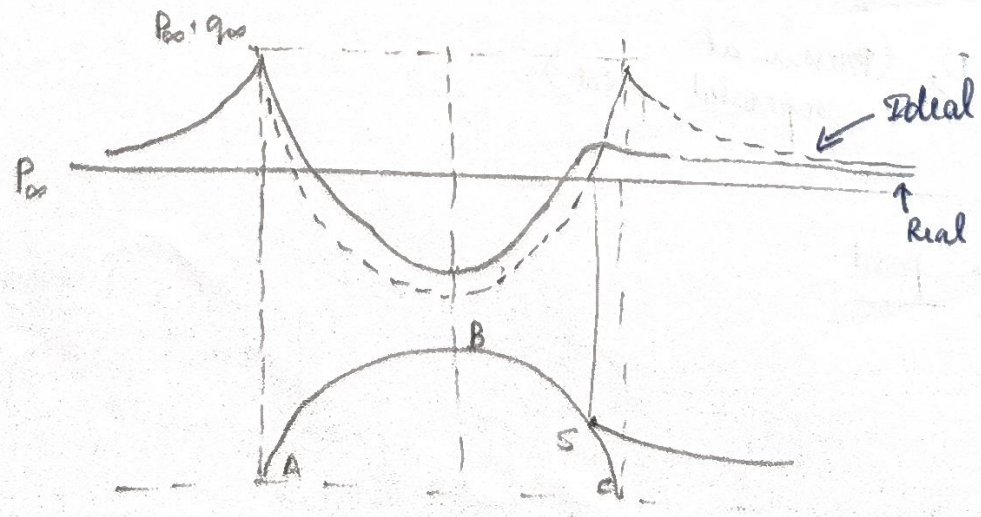
Pressure drag

In viscous flow, the displacement thickness increases in adverse pressure gradient



Viscous flow is necessary for pressure drag [even when there is no flow separation].

Consequence



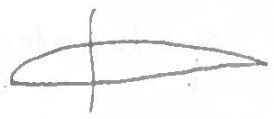
Profile drag:-

Friction + Pressure drag

Bluff body ⇒ Pressure drag > Friction drag

Streamline body ⇒ Friction > Pressure

In airfoil, 25-1. of geometry is used for



accelerating the flow. The other 3/4th is used to decelerate the flow

→ ~~Retard~~ Deceleration without loss is very difficult.

Friction drag

* depends on surface area
* almost independent of geometry

Pressure drag

* depends on geometry / body shape

Control of flow seperation

→ Suction

→ Blowing (must be tangential)

Classical BL (zeroth order)

Outer flow —Solution (Ue)→ Inner flow (BL).

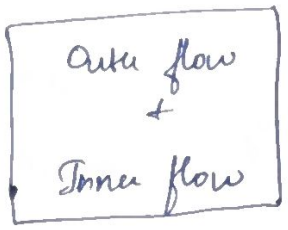
(BC, slip)[V=0]

Interacting BL (First order)

Outer flow —solution (Ue)→ Inner flow (BL)

(B.C : Ue, Ve)

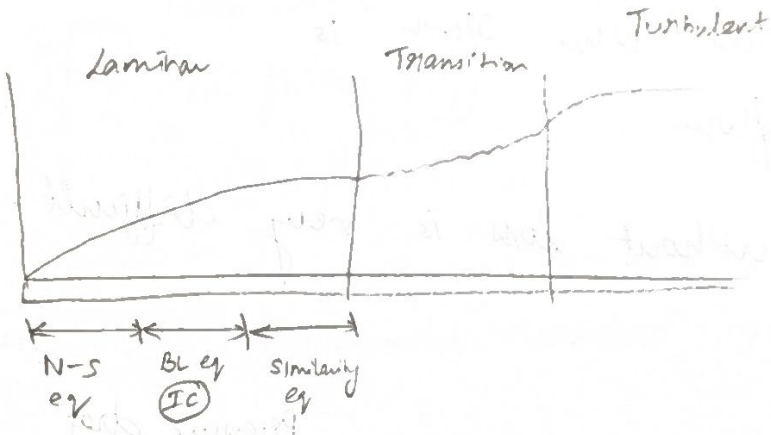
Simultaneous solution



BL

class.

Validity of boundary layer equation.



COMPRESSIBLE VISCOUS FLOW

Anderson, aerodynamics Ch-16.

→ Incompressible viscous flow : $\rho = const.$, $\mu = const$, $k = const.$
(const. property flow)

→ Compressible viscous flow : $\rho = f(P,T)$, $\mu \approx f(T)$, $k \approx f(T)$
(variable property flow)

Incompressible flow with heat transfer ($\mu = c$), $\rho = c$)

* mass balance
* momentum
* energy

} Convective H.T One way coupling $T \leftarrow u$).

Scenarios for compressible viscous flow

1) Heat transfer through solid walls.
2) High speed flows with adiabatic wall.

using equation of state. $\rho = \frac{P}{RT}$

$\mu(T)$ = Sutherland's relation

$K(T)$ =

$Pr = \frac{\mu C_p}{K} = \frac{\nu}{\alpha}$ $(T < 800K)$ $Pr \simeq 0.71$

C_p ≈ const. for air.

For gas : $\mu(T) \propto K(T)$

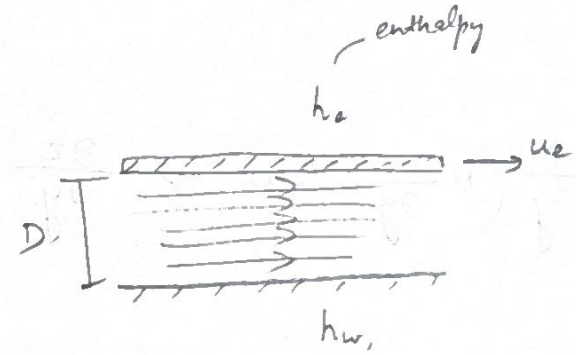
For liquids : $\times$

$Pr \approx 1$ gases

$Pr << 1$ - liquid, metals

$Pr >> 1$ - oils

Compressible Couette flow



Assumption

1) Plate is infinite
2) Steady
3) 2D

Boundary condition

@ $y = 0; u = 0, v = 0, h = h_w$

@ $y = D; u = u_e, v = 0, h = h_e$

Governing equation.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

Here. $\frac{\partial}{\partial x}(\cdot) = 0$ (Because infinit)

$$\Rightarrow \frac{\partial(\rho v)}{\partial y} = 0 \Rightarrow \boxed{\rho v = \text{const}}$$

Since $v = 0$ at $y = 0$. ; from above $\boxed{v = 0 \text{ everywhere}}$

x- momentum eqn

$$0 = \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y}\right]$$

y - momentum eqn $0 = \frac{\partial p}{\partial y}$

Energy (total) equation

$$0 = \frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y}\right]$$

If $\mu(T)$, $k(T)$, then analytical solution is not possible. Numerical solution is needed. (Assignment)

Constant property @ Couette flow -

* Temp variation is small enough to assume μ & k are constant.
* Temp. variation is sufficient enought to have heat transfer. ($T_w = T_e$)

G.F.

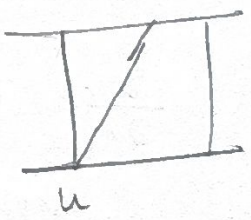
$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0 \Rightarrow \frac{\partial \tau}{\partial y} = 0 \Rightarrow \boxed{\tau = \text{const.}}$$ (momentum eqn)

$$\rightarrow \frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y}\right] = 0$$ (energy eqn)

Our objective : To find τ & $\dot{q}$.

x-mom : $\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow u = Ay + C$

$$\Rightarrow \boxed{u = \frac{U_e}{D} y}$$



Difference

Here, velocity profile is straight line. (constant property Couette flow)

In previous flow, it is not the case

Energy eqn :

$$\frac{\partial}{\partial y}\left[k\frac{\partial T}{\partial y} + \mu u \frac{\partial u}{\partial y}\right] = 0$$

$$k\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{2}\frac{\partial^2 u^2}{\partial y^2} = 0$$

$$\frac{1}{Pr}\frac{\partial^2 h}{\partial y^2} + \frac{1}{2}\frac{\partial^2 u^2}{\partial y^2} = 0$$

$$\frac{\partial^2 h}{\partial y^2} + \frac{Pr}{2}\frac{\partial^2 u^2}{\partial y^2} = 0$$

Integrating,

$$\boxed{h + Pr\frac{u^2}{2} = C_1 y + C_2}$$

Implementing boundary conditions, we get :-

$$C_2 = h_w$$

$$C_1 = \frac{h_e - h_w + \frac{Pr}{2}u_e^2}{D}$$

Solving, we get final expression.

$$\boxed{h = h_w + \left[h_e - h_w + \frac{Pr}{2}u_e^2\right]\frac{y}{D} - \frac{Pr}{2}u_e^2\left(\frac{y}{D}\right)^2}$$

Heat flux.

$$\dot{q} = -k\frac{\partial T}{\partial y} = -\frac{k}{c_p}\frac{\partial h}{\partial y} = -\frac{\mu}{Pr}\left(\frac{\partial h}{\partial y}\right)$$

$$\dot{q} = -\frac{\mu}{Pr}\left[h_e - h_w + \frac{Pr}{2}u_e^2\right]\frac{1}{D} + \mu u_e^2\frac{y}{D^2}$$

$$\boxed{\dot{q} = -\frac{\mu}{Pr}\left(\frac{h_e - h_w}{D}\right) + \mu\frac{u_e^2}{D^2}\left(y - \frac{D}{2}\right)}$$

Here,

$$\mu\frac{u_e^2}{D^2} = \mu\left(\frac{u_e}{D}\right)\left(\frac{u_e}{D}\right) = \tau\frac{\partial u}{\partial y} = \Phi$$

← Dissipation

Heat transfer in the wall

@ $y = 0$,

$$\dot{q}_{w,y=0} = \frac{-\mu}{Pr}\left[\frac{(h_e - h_w) + \frac{Pr}{2} u_e^2}{D}\right]$$

@ $y = D$,

$$\dot{q}_{w,y=D} = \frac{-\mu}{Pr}\left[\frac{(h_e - h_w) - \frac{Pr}{2} u_e^2}{D}\right]$$

Three cases are looked into:

* negligible viscous dissipation
* Equal wall temperature
* Adiabatic wall.

1) Negligible viscous dissipation

when u_e is small. $\Longrightarrow$ Neglect u_e^2 term.

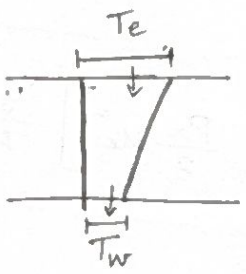
$$E_c = \frac{u}{c_p \Delta T}$$

$E_c \to 0$

$E_c \propto M^2$

$$h = h_w + (h_e - h_w)\frac{y}{D}$$

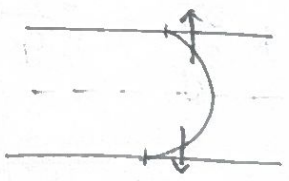
$$\dot{q}(y) = -\frac{\mu}{Pr}\left(\frac{h_e - h_w}{D}\right) = \text{const.}$$



2) Equal wall temperature

$$h_e = h_w$$

$$h = h_w + \left(\frac{Pr}{2} u_e^2\right)\frac{y}{D} - \frac{Pr}{2} u_e^2 \left(\frac{y}{D}\right)^2$$



Here, temp. of liquid is higher than wall temp. This is due to viscous effects

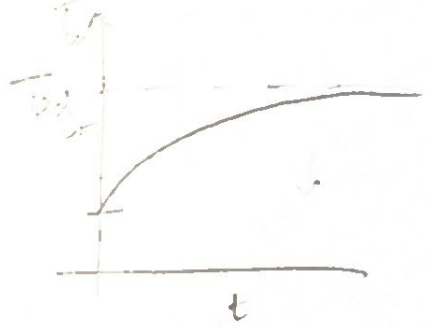
Viscous dissipation is causing increase in temperature of flow field.

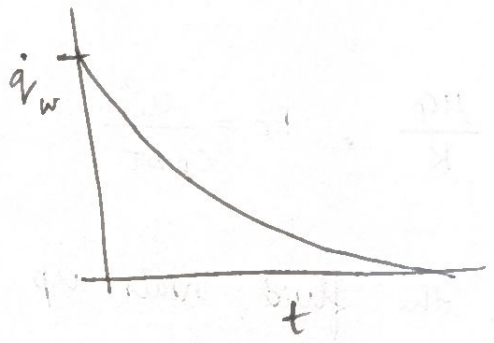
External heat removal mechanism maintains constant $h_e = h_w = \text{const.}$

3) Adiabatic wall (lower)

With ↓ heat removal mechanism from previous is not there ?!

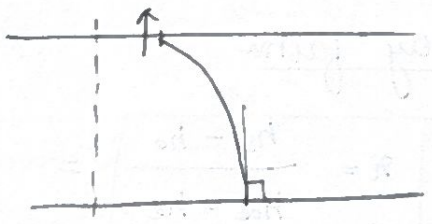
T_w will increase.





This asymptotic temperature is called adiabatic wall temperature.

$$\dot{q} = 0 = \left.\frac{\partial h}{\partial y}\right|_w = 0 \quad \Rightarrow \quad h_w = h_{aw}$$



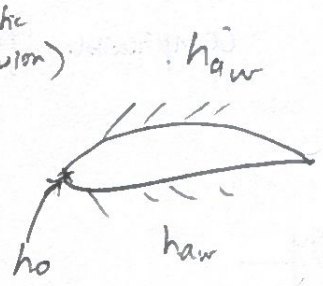
$$\Rightarrow \boxed{h_{aw} = h_e + Pr\,\frac{u_e^2}{2}}$$

For this problem ↑

or (in general)

$$\boxed{h_{aw} = h_e + r\,\frac{u_e^2}{2}}$$

But we know that, $h_o = h_e + \frac{u_e^2}{2}$, (Adiabatic compression)



comparing, $h_o \geq h_{aw}$

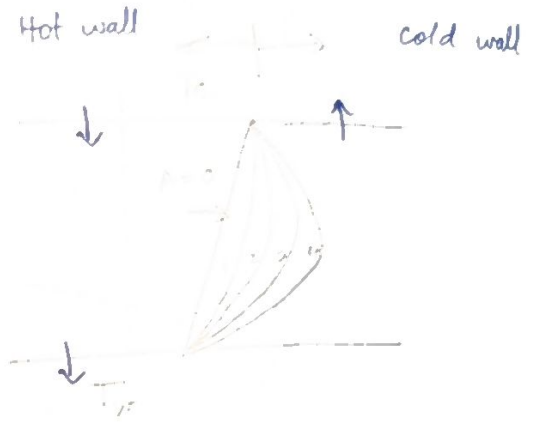
Both are reached when $v = 0$.

* Adiabatic wall temp. is the temp. of surface perfectly insulated on its back side will reach due to viscous dissipation

6/4/26

$T_e > T_w$, viscous heating

$$\frac{T - T_w}{T_e - T_w} = \frac{y}{D} + \underbrace{\frac{Pr\,Ec}{2}}_{A} \frac{y}{D}\left(1 - \frac{y}{D}\right)$$



Here, $Pr = \frac{\mu C_p}{K}$, $Ec = \frac{u_e^2}{C_p \Delta T}$

* As flow takes place, the fluid heats up due to viscous heating, and direction of heat transfer changes.

(A ≈ 0 for very small velocities).

Recovery factor

$$r = \frac{h_r - h_e}{h_{oe} - h_e} = Pr$$ ← (For Couette flow [constant prop.])

* In general, incompressible flow, $r = f(Pr)$

compressible flow, $r = f(Pr, M, \gamma)$

Reynold's analogy

In experiments, it is difficult to measure heat transfer or temp. profile.

But, easier to obtain shear stress.

Q. Can we get heat transfer given shear stress.

* Relation between shear stress coefficient & heat transfer coefficient.

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$

Stanton number (C_h, St)

$$C_h = \frac{\text{Heat transfer coefficient}}{\text{Heat capacity of the fluid} / \text{Area} \times \text{time}}$$

$$= \frac{\dot{q}/\Delta T}{\rho u C_p}$$

$$C_h = \frac{Nu}{Re} Pr \qquad \left(Nu = \frac{hL}{k}\right)$$

For constant property Couette flow

$$C_h = \frac{\frac{\mu}{Pr}\frac{(h_{aw} - h_e)}{D}}{\rho_e u_e (h_{aw} - h_e)} \Rightarrow \boxed{C_h = \frac{1}{Re\, Pr}} \leftarrow \text{(For specific flow)}$$

$$C_f = \frac{2}{Re} \Leftarrow \left[\frac{\mu\left(\frac{u_e}{D}\right)}{\frac{1}{2}\rho_e u_e^2}\right]$$

Now, $\boxed{\frac{C_h}{C_f} = \frac{1}{2}\frac{1}{Pr}} = f(Pr)$

(Assignment)

COMPRESSIBLE BOUNDARY LAYER

* Thin shear layer approximation

$$\frac{\delta}{x} << 1, \quad \frac{v}{u} << 1, \quad Re >> 1$$

Local scale: $x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}\sqrt{Re}$

$$u^* = \frac{u}{u_e}, \quad v^* = \frac{v}{u_e}\sqrt{Re}$$

$$\theta = \frac{T - T_e}{\Delta T_{ref}}$$

As $Re \to \infty$.

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{dp_e}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$0 = \frac{\partial p}{\partial y}$$

$$\rho\left[u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right] = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + u\frac{dp_e}{dx} + \mu\left(\frac{\partial u}{\partial y}\right)^2$$

$$\Rightarrow \rho\left[u\frac{\partial h_o}{\partial x} + v\frac{\partial h_o}{\partial y}\right] = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right)$$

Here, x-derivatives vanish.

Compressible Couette Flow

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(\mu u\frac{\partial u}{\partial y}\right) = 0$$

Non-dimensional form

$$\frac{\partial}{\partial y'}\left(\mu'\frac{\partial u'}{\partial y'}\right) = 0$$

$$\frac{\partial}{\partial y'}\left(k'\frac{\partial T'}{\partial y'}\right) + Pr\,Ec\frac{\partial}{\partial y'}\left(\mu' u'\frac{\partial u'}{\partial y'}\right) = 0$$

$$Ec = \frac{u^2}{c_p T} = (\gamma - 1)M^2$$

$$Pr = \frac{\mu c_p}{k} \Rightarrow \frac{\mu(T)\, c_p}{k(T)}.$$

c_p = const. & Pr = const.

$$k(T) = \frac{\mu(T) c_p}{Pr}.$$ Anderson, Aerodynamics.

Unknowns : $\rho, u, v, h, T, c_p, k, \mu$ — 8 unknowns.

Equations : ③ from previous

+ TEOS $\Rightarrow$ $P = \rho RT$

+ CEOS $\rightarrow$ $e(T)$, $h(T)$

$\mu(T)$

$k(T)$

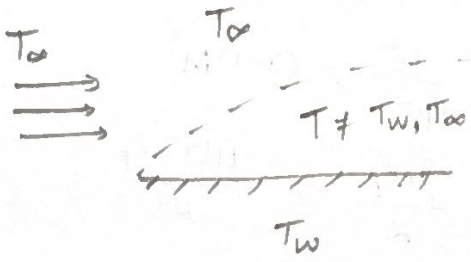
$c_p(T)$

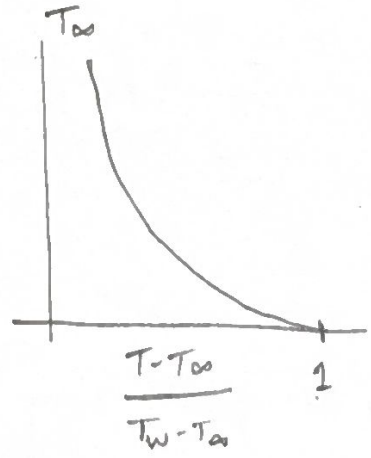
8 equations

Here, $\frac{dP_e}{dx}$ is the input, to be obtained from inviscid outer flow.

Euler eqn: $-\frac{dp_e}{dx} = \rho_e u_e \frac{du_e}{dx}$.

Thermal Boundary Layer





$\delta_T \propto \sqrt{(\text{diffusivity}) \times \text{time}}$

$\delta_T \propto \sqrt{\alpha t}$

$\sqrt{Pr} = \sqrt{\frac{\mu c_p}{k}} = \sqrt{\frac{\nu}{\alpha}} = \frac{\delta_u}{\delta_T}$

$Pr \gg 1$ oils $\quad$ $Pr \ll 1$ Liquid metals

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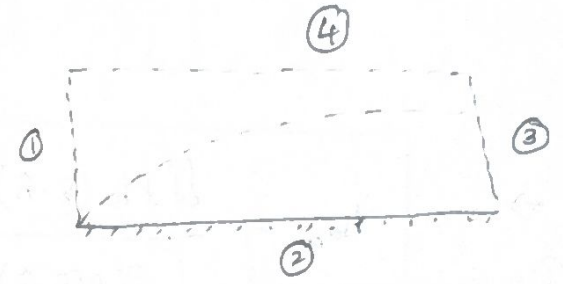
CONTROL VOLUME ANALYSIS OF ADIABATIC COMPRESSIBLE BL

Assumption

1) steady
2) Adiabatic wall
3) 2D



Energy equation

$$\oiint \rho h_o (\underline{v}\cdot\hat{n})\, dA = -\oiint \underline{\bar{q}}\cdot\hat{n}\, dA + \iint \underline{\underline{\tau}}\cdot\underline{u}\,\hat{n}\, dA \qquad \text{—— (1)}$$

Mass

$$\oiint \rho \underline{v}\,\hat{n} dA = 0 \quad \Rightarrow \quad \oiint \rho h_{o_\infty} \underline{v}\,\hat{n} dA = 0 \qquad \text{—— (2)}$$

BL approximation

[x- derivative is zero]

$$\underline{\underline{\tau}} = \begin{bmatrix} 0 & \tau_{21} \\ \tau_{21} & 0 \end{bmatrix} \qquad \underline{\bar{q}} = \begin{Bmatrix} 0 \\ q_{ij} \end{Bmatrix} \qquad v = 0$$

Now, ① − ②

A B

$$\Rightarrow \boxed{\oiint \rho (h_o - h_{o_\infty})\, \underline{v}\cdot\hat{n}\, dA = -\oiint \underline{q}\cdot\hat{n}\, dA + \iint \underline{\underline{\tau}}\cdot\underline{u}\,\hat{n}\, dA} \qquad \text{—— (3)}$$

A

$$= S_1 + S_2 + S_3 + S_4 = 0 + 0 + \begin{Bmatrix} 0 \\ q_y \end{Bmatrix} \cdot \begin{Bmatrix} n_x \\ 0 \end{Bmatrix} + 0$$

($q_y \to 0$)

B

$$= S_1 + S_2 + S_3 + S_4 = \underset{(\underline{\underline{\tau}}=0)}{0} + \underset{(\underline{u}=0)}{0} + \begin{bmatrix} 0 & \tau_{21} \\ \tau_{21} & 0 \end{bmatrix} \begin{Bmatrix} u \\ 0 \end{Bmatrix} \begin{Bmatrix} n_x \\ 0 \end{Bmatrix} + 0$$

($\tau_{21} \to 0$)

Thus, $\boxed{RHS = 0}$.

LHS

$$= S_1 + S_2 + S_3 + S_4 = \underset{(h_o = h_\infty)}{0} + \underset{(\underline{v}=0)}{0} + S_3 + \underset{(h_o = h_\infty)}{0}$$

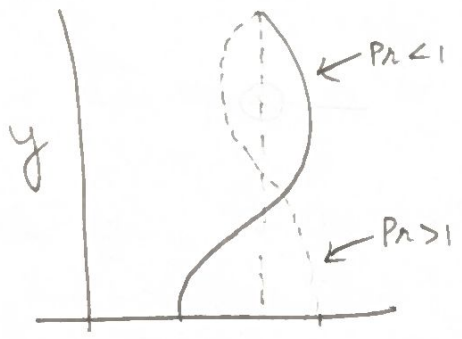
$$= \iint_{S_3} \rho (h_o - h_{o_\infty})\, \underline{v}\cdot\hat{n}\, dA = 0$$

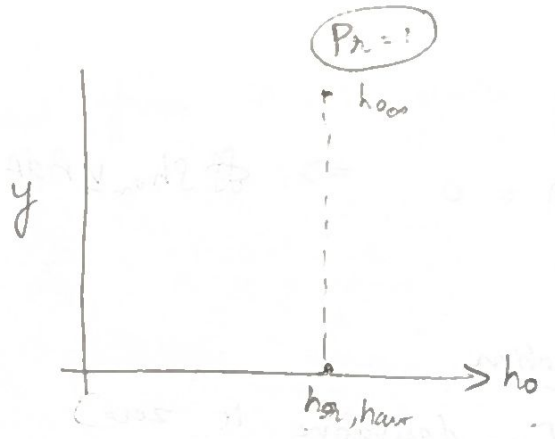
LHS $\Rightarrow$ $\iint_{A_3} \rho h_o (\underline{v}\cdot\hat{n})\, dA = \iint_{A_3} \rho h_{o\infty} (\underline{v}\cdot\hat{n})\, dA$

$= h_{o\infty} \iint \rho (\underline{v}\cdot\hat{n})\, dA$

$\Rightarrow$ $$h_{o\infty} = \frac{\iint \rho h_o (\bar{v}\cdot\hat{n})\, dA}{\iint \rho (\bar{v}\cdot\hat{n})\, dA} = \bar{h_o}$$

Total enthalpy along a cross-section in BL at any point should be equal to the free stream total enthalpy





This phenomena is due to ~~to~~ condition imposed by the above formula. The average total enthalpy should be same. Thus, if it is lower at some place, it compensates for it.

Coupled BL : velocity field linked with Temperature field.

Energy eq in differential (enthalpy)

$$\rho \frac{Dh_o}{Dt} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j}\right) + \frac{\partial}{\partial x_j}(\tau_{ij} u_i)$$

BL approximation (steady)

$$\rho \left[u \frac{\partial h_o}{\partial x} + v \frac{\partial h_o}{\partial y}\right] = \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial y}\left(u \mu \frac{\partial u}{\partial y}\right) \quad \text{——— (i)}$$

A

$$\textcircled{A} = \frac{\partial}{\partial y}\left(\frac{k}{c_p}\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) \qquad \left[\text{Using } h_0 = h + \tfrac{1}{2}u^2\right]$$

$$= \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial h_0}{\partial y}\right] - \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\,u\frac{\partial u}{\partial y}\right]$$

Thus eq (i) becomes,

$$\rho\left[u\frac{\partial h_0}{\partial x} + v\frac{\partial h_0}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\mu}{Pr}\frac{\partial h_0}{\partial y}\right] + \frac{\partial}{\partial y}\left[\left(1-\frac{1}{Pr}\right)\mu\, u\frac{\partial u}{\partial y}\right] \qquad \longrightarrow (1)$$

Energy Integral / Crocco-Busemann relations

Case I $Pr = 1$ & Adiabatic wall

$$\rho\left[u\frac{\partial h_0}{\partial x} + v\frac{\partial h_0}{\partial y}\right] = \frac{\partial}{\partial y}\left[\mu\frac{\partial h_0}{\partial y}\right]$$

Boundary conditions:

@ $y=0$, $u=0$; $\frac{\partial T}{\partial y} = \frac{\partial h}{\partial y} = \frac{\partial h_0}{\partial y} = 0$

@ $y=\infty$, $u=u_e$, $h_0 = h_{0e}$

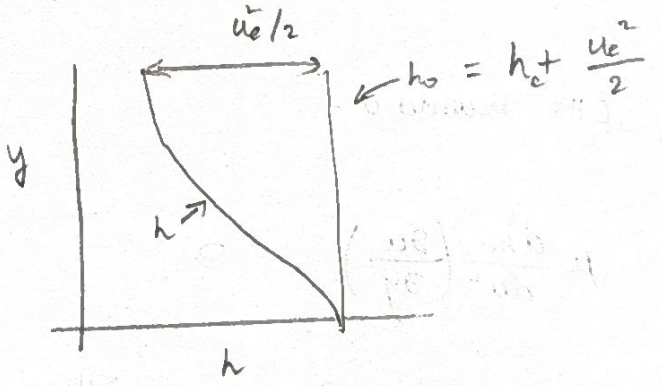
h_0 = constant is also a solution for this.

$$h_0 = h_{0e} = h + \frac{u_e^2}{2} = h_e + \frac{u_e^2}{2}$$

$$\Rightarrow \quad h = h_e + \frac{u_e^2}{2}\left[1-\left(\frac{u}{u_e}\right)^2\right]$$

u_e is known from inviscid solution

$$h = f(u)$$



Calorically perfect gas

$$\frac{T}{T_0} = 1 + \frac{\gamma-1}{2}M_e^2\left[1-\left(\frac{u}{u_e}\right)^2\right]$$

* Result is indipendent of pressure gradient.

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Case II. $Pr = 1$ & const. temperature with $\frac{dp}{dx} = 0$

$$\rho\left[u\frac{\partial h_0}{\partial x} + v\frac{\partial h_0}{\partial y}\right] = \frac{\partial}{\partial y}\left[\mu\frac{\partial h_0}{\partial y}\right] \qquad \longrightarrow (2)$$

Momentum eq

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \cancel{\frac{\partial p_e}{\partial x}}^{\,0} + \frac{\partial}{\partial y}\left[\mu\frac{\partial u}{\partial y}\right] \qquad \longrightarrow (3)$$

Comparing both, they seem same

Boundary condition:

@ $y = 0$, $u = 0$, ~~$\theta = 0$~~ @ $y = 0$, $h_o = h_w$

@ $y \to \infty$, $u = u_e$ @ $y \to \infty$: $h_o = h_{oe}$

From mathematics, these type of equations imply that

$h_o = h_o(u)$ → linear fn. of u.

$h_o(u)$, $u(x,y)$.

$$\therefore \frac{dh_o}{dx} = \frac{dh_o}{du}\frac{\partial u}{\partial x} \quad \Big| \quad \frac{dh_o}{dy} = \frac{dh_o}{du}\frac{\partial u}{\partial y}.$$

Substitute into eq ②,

$$\Rightarrow \rho \frac{dh_o}{du}\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial y}\left[\mu \frac{dh_o}{du}\frac{\partial u}{\partial y}\right]$$

$$= \frac{dh_o}{du}\frac{\partial}{\partial y}\left[\mu\frac{\partial u}{\partial y}\right] + \mu\frac{\partial u}{\partial y}\frac{d^2h_o}{du^2}\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{dh_o}{du}\left(\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right]\right) - \frac{dh_o}{du}\frac{\partial}{\partial y}\left[\mu\frac{\partial u}{\partial y}\right] = \mu\frac{d^2h_o}{du^2}\left(\frac{\partial u}{\partial y}\right)^2$$

From eq ③, LHS becomes 0.

$$\Rightarrow \mu\frac{d^2h_o}{du^2}\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

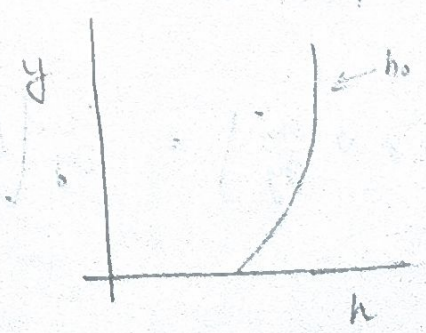
Here, $\mu \neq 0$ & $\frac{\partial u}{\partial y} \neq 0$. Thus

$$\boxed{\frac{d^2h_o}{du^2} = 0}$$

Solving for h_o,

$$\Rightarrow h_o = (h_{oe} - h_w)\frac{u}{u_e} + h_w$$

$$\Rightarrow \boxed{\frac{h_o - h_w}{h_{oe} - h_w} = \frac{u}{u_e}}$$



$$h = (h_{oe} - h_w)\frac{u}{u_e} + h_w - \frac{u^2}{2} \qquad \text{(Static enthalpy)}$$

$$T = (T_{oe} - T_w)\frac{u}{u_e} + T_w - \frac{u^2}{2c_p} \qquad \text{(Temperature)}$$

$$\Rightarrow \frac{T - T_w}{T_e} = \frac{T_{oe} - T_w}{T_e}\left(\frac{u}{u_e}\right) - \frac{\gamma-1}{2} M_e^2 \left(\frac{u}{u_e}\right) \qquad \text{[Non-dimensional]}$$

Thus, $\frac{T}{T_e} = f\left(Pr, M, \gamma, \frac{T_w}{T_e}\right)$

Reynolds analogy.

$$C_h = \frac{q_w}{(h_{oe} - h_w)\rho_e u_e} \qquad C_f = \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2}$$

$$q_w = \frac{-\mu}{Pr}\frac{\partial h_o}{\partial y} = \frac{-\mu}{Pr}\frac{1}{u_e}\frac{\partial u}{\partial y}(h_{oe} - h_w)$$

Now, $\frac{C_h}{C_f} = \frac{1}{2}$ | $Pr = 1$, $\frac{dP_e}{dx} = 0$

In general, $\frac{C_h}{C_f} = \frac{1}{2\,Pr^{2/3}}$.

Heat flux.

$$\dot{q} = -k\frac{\partial T}{\partial y} = -\frac{k}{u_e}\left[(T_{oe} - T_w) - \frac{\gamma-1}{2} M_e^2 T_e \left(\frac{u}{u_e}\right)\right]\left(\frac{\partial u}{\partial y}\right)$$

$$\dot{q} = \frac{c_p \tau}{u_e Pr}\left[(T_w - T_{oe}) + (\gamma-1) M_e^2 T_e \left(\frac{u}{u_e}\right)\right]$$

At wall, $(u=0)$

$$\dot{q}_w = \frac{c_p \tau}{u_e}(T_w - T_{aw})$$

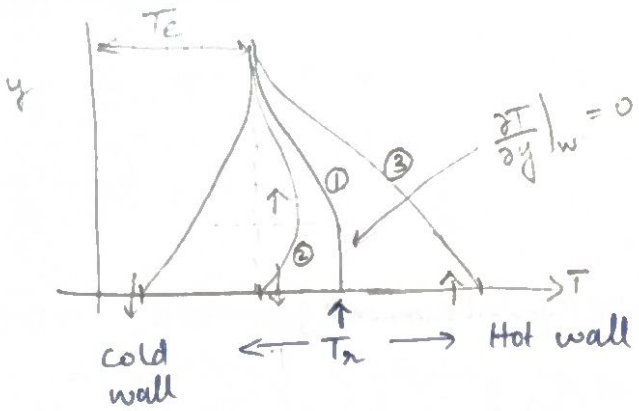
Inside BL, $\tau > 0$.

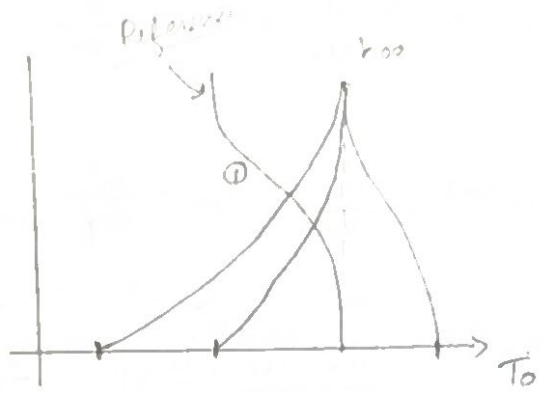
Heat transfer to wall depends on $(T_w - T_{aw})$ not $(T_w - T_e)$ ✗.

Case (i) : $T_w = T_{aw}$: $\dot{q}_w = 0$

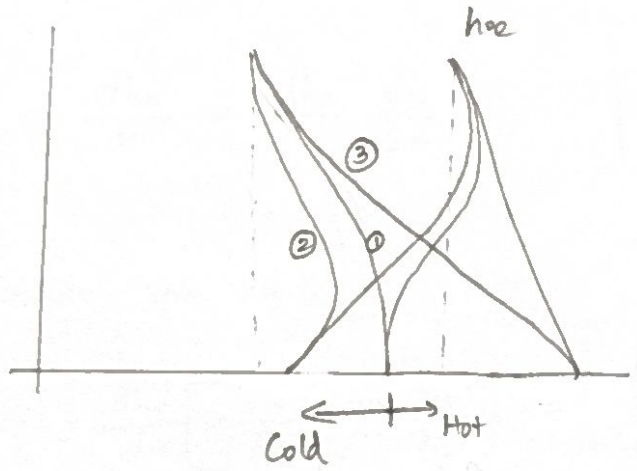
Case (ii) : $T_w > T_{aw}$: $\dot{q}_w > 0$, $\left.\frac{\partial T}{\partial y}\right|_w < 0$

Case (iii) : $T_w < T_{aw}$: $\dot{q}_w < 0$, $\left.\frac{\partial T}{\partial y}\right|_w > 0$





$Pr < 1$



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Compressible BL

Even incompressible BL is difficult to solve due to non-linear nature.

Transform (purpose)

1) Similarity solution
2) Numerical solution
3) To reduce the eqn into incomp. BL eqn.

Ref :- Anderson Hypersonic flow

Independent variable

$$(x, y) \Rightarrow (\xi, \eta)$$

$$\xi = \int_0^x \rho_e(x)\, u_e(x)\, \mu_e(x)\, dx$$

$$\eta = \frac{u_e}{\sqrt{2\xi}} \int_0^y \rho\, dy$$

Dependent variable

$$f' = \frac{u}{u_e}$$

$$g = \frac{h}{h_e}$$

By using this Transformation, momentum & energy eqn. can be derived.
(Reference)

C → Chapman - Rubesin factor.

If C = 1 , one way coupling

Adiabatic wall

Mach number increases, BL thickness increases

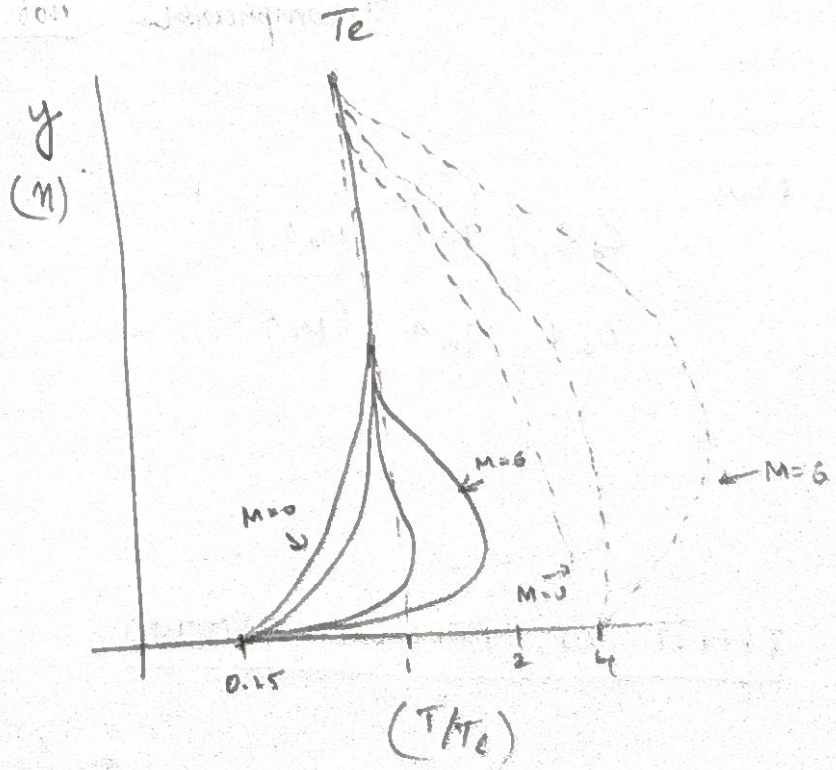
$M_e \uparrow, T \uparrow \Rightarrow \delta \downarrow \Rightarrow \delta_u \uparrow$ velocity BL

Aerodynamic heating

- Adiabatic compression (Stagnation point) (inviscid).
- Viscous heating

Cold wall BL temp. < Adiabatic wall BL temp.

* Heat transfer in BL is dependent on wall temp. & flow mach number.



C_f (vs) Me

$C_f \propto \tau_w \propto \mu \left(\frac{\partial u}{\partial y}\right)$

$M_e \uparrow \Rightarrow \delta_u \uparrow \Rightarrow \frac{\partial u}{\partial y} \downarrow \Rightarrow C_f \downarrow$

But, ~~$\tau_w \uparrow$ M~~

$M_e \uparrow \; \tau_w \uparrow$

From graph,

$\boxed{\eta \approx \sqrt{Pr}}$ $\qquad$ $\boxed{\frac{C_h}{C_f} \propto Pr^{-2/3}}$

$T \uparrow \Rightarrow \rho \downarrow \Rightarrow \delta_u \uparrow$

$\downarrow$

$\uparrow T_w$ or $M_e \uparrow$

$\dot{q}_w \propto (h_{aw} - h_w)$

Class

Incompressible	Compressible
$\frac{\delta}{x} = \frac{5}{\sqrt{Re_x}}$	$\frac{\delta}{x} = \frac{F(M_e, Pr, \gamma, T_w/T_e)}{\sqrt{Re_x}}$
$C_f = \frac{0.664}{\sqrt{Re_x}}$	$C_f = \frac{G_1(M_e, Pr, \gamma, T_w/T_e)}{\sqrt{Re_x}}$
$C_h = \frac{0.332}{\sqrt{Re_x}} Pr^{-2/3}$	$C_h = \frac{G_2(M_e, Pr, \gamma, T_w/T_e)}{\sqrt{Re_x}}$
$\frac{C_h}{C_f} = \frac{1}{2} Pr^{-2/3}$	$\frac{C_h}{C_f} = f(M, Pr, \gamma, T_w/T_e)$

→ For both incompressible & compressible, length scale dependence is same, $\sqrt{x}$.

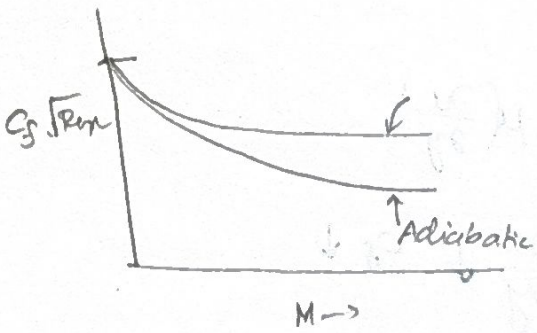
→ velocity dependence

Incompressible not same as compressible flow.

Now, Drag

$C_f \downarrow$, $D \uparrow$ $(M_e \uparrow)$

$C_h \downarrow$, $q_w \uparrow$ $(M_e \uparrow)$.



EFFECT OF PRESSURE GRADIENT

Incompressible : APG ⇒ $\delta_u \uparrow$

FPG ⇒ $\delta_u \downarrow$

Compressible : Same

Flow seperation

1) Viscous

2) Adverse pressure gradient

Effect of Mach number

$M\uparrow$, $T\uparrow \Rightarrow \delta_u \uparrow$

Flow seperation occurs early.

* Even though high mach number (curve is outside) then high temp. → this leads to lower velocity.

Heating the wall (T_w)

$T_w \uparrow \; \delta_u \uparrow$ Flow seperation occurs early.

Reference Temperature Method

Can we use the relation of C_f in inviscid to find property in compressible flow?

$$C_f = \frac{0.664}{\sqrt{Re_x}}$$

$$Re_x^* = \frac{\rho^* U_e x}{\mu^*} \quad \rho^*(T^*),\ \mu^*(T^*).$$

(compressible) $$C_f^* = \frac{0.664}{\sqrt{Re_x^*}}$$

It was found that (approximation) :-

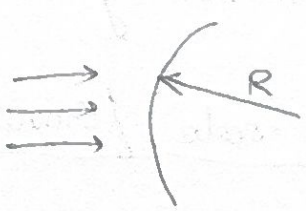
$$\frac{T^*}{T_e} = 1 + 0.032\, Me^2 + 0.58\left(\frac{T_w}{T_e} - 1\right)$$

$$C_h^* = \frac{0.332}{\sqrt{Re^*}} (Pr^*)^{-2/3}$$

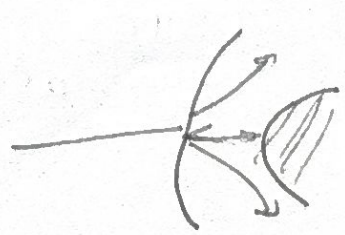
Stagnation point BL

→ Similarity solution.

$$\dot{q}_w \propto \frac{1}{\sqrt{R}}$$



(viscous heating)



The reason for blunt body in heat dissipation is that bow shock is detatched & far away, more air take away heat

$$C_h = \frac{q_w}{\rho_e u_e (h_{aw} - h_w)}$$

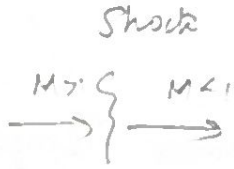
HCU, $h_{aw} - h_w = \frac{r}{2}\frac{u_e^2}{2} \approx u_e^2$ (High u_e)

$$\Rightarrow \boxed{\dot{q}_w = \rho_e u_e^3 C_h} \quad \& \quad \boxed{D = \frac{1}{2}\rho_e u_e^2 C_f}$$

Nature of behaviour in subsonic & supersonic flow

Inviscid.

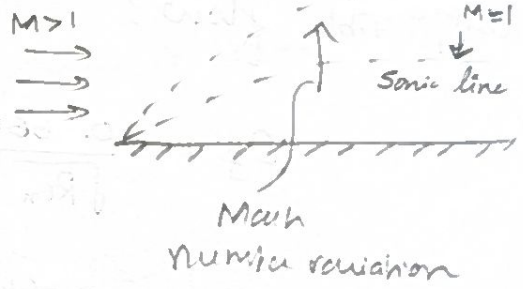
Drastic qualitative change from subsonic & supersonic flow.



Viscous.

Not much qualitative change (only diffusion).

No shock is formed inside BL due to flow regime.



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TURBULENT FLOWS

Turbulence

→ Random like (or) chaotic, disorganised (always unsteady)

→ Non-repeatability (sensitive to initial condition). → comes from non-linearity

→ Multi-scale flow.

→ Enhanced mixing & dissipation.

→ 3D, time dependence & vorticity.

→ Intermittency in both space & time

Multi-scale / Generation of scales.

1D.

Advection: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

IC

$u(x, t_0) = A\cos(k_x x)$.

k_x – wave number

Time

Period - T

Freq $= \frac{1}{T}$

Angular frequency $= \frac{2\pi}{T}$

Space

~~Rz~~ · wavelength $= \lambda$

$-$ $1/\lambda$

wave number $= \frac{2\pi}{\lambda}$

$k_x = \frac{2\pi}{\lambda_x}$.

$$u(t_0+\Delta t) = u(t_0) - \Delta t \left.\frac{\partial u}{\partial t}\right|_{t_0}$$

$$= A\cos(k_n x) + \Delta t \left[-u \frac{\partial u}{\partial x}\right]_{t_0}$$

$$= A\cos(k_x x) + \Delta t \left[A\cos(k_n x)\left(A k_x \sin k_n x\right)\right]$$

$$u(t_0+\Delta t) = A\cos(k_n x) + \Delta t \frac{A^2 k_x}{2} \sin(2k_n x).$$

Initial : $k_n \longrightarrow k_n, 2k_n$ — Higher harmonics.

2D.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

IC.

$$u(x,y,t_0) = A\cos(k_x x)\cos(k_y y)$$

$$u(x,y,t_0) = A'\cos(k'_x x)\cos(k'_y y)$$

HW.

⇒ Initial : k_n, k_y, k'_n, k'_y $\xRightarrow{\Delta t}$ $2k$, $k_n + k'_n$, $k_y + k'_y$ (Higher harmonics)

$-k_n - k'_n$, $k_y - k'_y$ (Lower harmonics)

Statistics

Ensemble - Collection of independent realization of a random variable obtained under identical conditions.

Ensemble average

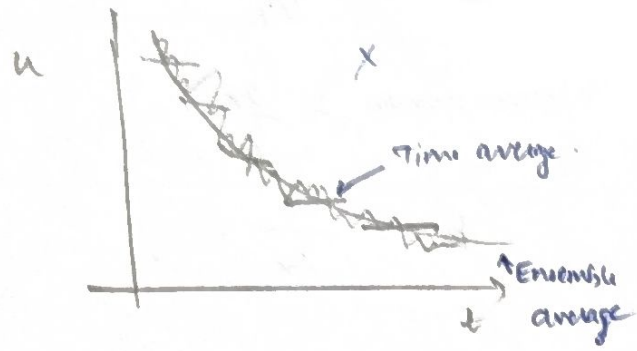
$$\bar{u}(x) = \frac{1}{N}\sum_{n=1}^{N} u(x,t,n)$$

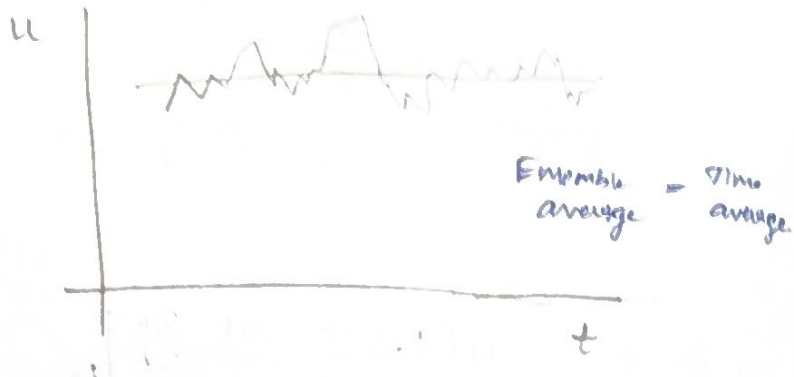
Time average.

$$\bar{u}(x) = \frac{1}{\Delta t}\int_{t_0}^{t_0+\Delta t} u(x,t)\,dt$$

Stationary in time (Not STEADY)

→ Statistics are independent of time





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BOUNDARY LAYER THEORY

Turbulence

Turbulence is governed by deterministic eqns. → Thus, random-like.

Properties of ensemble average.

* Commutes with differentiation, integration, addition & subtraction.

$$f = \bar{F} + f'$$

(f: instantaneous; $\bar{F}$: mean quantity; f': Fluctuation)

Average, $\bar{f} = \overline{\bar{F} + f'} = \bar{F} + \bar{f'}^{\,0} = \bar{F}$

Now, $\overline{f+g} = \bar{f} + \bar{g}$

$\overline{\alpha g} = \alpha \bar{g}$

$\overline{\bar{F} g} = \bar{F}\bar{g} = \bar{F}\bar{g}$

$\overline{\frac{\partial f}{\partial x}} = \frac{\partial \bar{f}}{\partial x}$ & $\overline{\int f\,dx} = \int \bar{f}\,dx$.

$\overline{f^m} \neq (\bar{f})^m$

$\overline{fg} = \overline{(\bar{f}+f')(\bar{g}+g')} = \overline{\bar{f}\bar{g} + f'\bar{g} + \bar{f}g' + f'g'} = \bar{f}\bar{g} + \overline{f'}^{\,0}\bar{g} + \bar{f}\overline{g'}^{\,0} + \overline{f'g'}$

$$\boxed{\overline{fg} = \bar{f}\bar{g} + \overline{f'g'}}$$

Moments

$$\overline{u^m(x,t)} = \frac{1}{N}\sum_{n=1}^{N} u^m(x,t;n)$$

If m = integer ⇒ Statistics

If m = 1 ⇒ Mean / Average

Central Moments (To remove the mean)

$(\underbrace{u - \bar{u}}_{\text{fluctuation}})^m$ = m^{th} central moment.

First central moment

$$\overline{(u-\bar{u})} = 0$$

Second central moment

$$\overline{(u-\bar{u})^2} = \text{Variance}$$

Third central moment

$$\overline{(u-\bar{u})^3} = \text{Skewness}$$

Fourth central moment

$$\overline{(u-\bar{u})^4} = \text{Kurtosis}.$$

* Standard deviation $= \sqrt{\text{Variance}} = \sqrt{\overline{(u-\bar{u})^2}}$

* RMS = Root mean square

$$= \sqrt{\overline{u^2}}$$

Reynolds Average Navier Stokes eqn (RANS)

Objective. To get mean equation.

1) Reynoldi decomposition $q = \bar{q} + q'$

2) Average (Ensemble).

Incompressible N-S eqn

Mass conservation $\frac{\partial u_i}{\partial x_j} = 0$ $\quad u_i = \bar{u_i} + u_i' \Rightarrow \frac{\partial \overline{[\bar{u_i} + u_j]}}{\partial x_j}$

$$\Rightarrow \boxed{\frac{\partial \bar{u_i}}{\partial x_j} + 0 = 0} \qquad \frac{\partial u_i'}{\partial x_i} = 0 \quad \text{— (A)}$$

Momentum eqn.

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j} \qquad \rho \to \text{constant}$$

Reynolds Averaging

$$\Rightarrow \frac{\partial(\rho u_i)}{\partial t} \overset{\uparrow}{=} \frac{\partial}{\partial t}\overline{[\rho(\bar{u_i} + u_i')]} = \frac{\partial}{\partial t}(\rho \bar{u_i})$$

Illy

$$\Rightarrow \frac{\partial P}{\partial x_i} \overset{RA}{=} \frac{\partial \bar{P}}{\partial x_i} \quad , \quad \Rightarrow \frac{\partial \tau_{ji}}{\partial x_j} \overset{RA}{=} \frac{\partial \bar{\tau}_{ji}}{\partial x_j}$$

$$\Rightarrow \frac{\partial}{\partial x_j}[\rho u_i u_j] \overset{RA}{=} \frac{\partial}{\partial x_j}\left[\rho \bar{u_i}\bar{u_j} + \overline{\rho u_i' u_j'}\right]$$

Assembling the equations,

$$\frac{\partial}{\partial t}(\rho \bar{u}_i) + \frac{\partial}{\partial x_j}(\rho \bar{u}_i \bar{u}_j) = \frac{\partial}{\partial x_j}\left[-\bar{p}\delta_{ij} + \tau_{ij} - \underbrace{\rho \overline{u_i' u_j'}}_{\text{extra}}\right]$$

$$\rho \frac{\bar{D}}{Dt}(\bar{u}_i) = \frac{\partial}{\partial x_j}\left[\underbrace{-\bar{p}\delta_{ij}}_{\text{Isotropic stress}} + \underbrace{\bar{\tau}_{ij}}_{\text{Viscous}} - \underbrace{\rho \overline{u_i' u_j'}}_{\text{Apparent stress due to velocity fluctuation.}}\right] \quad \text{——— (B)}$$

Momentum transfer due to fluctuating velocity field

Here, $-\rho \overline{u_i' u_j'}$ → apparent stress

$\overline{u_i' u_j'}$ → Reynold's stress.

(A) & (B) together form RANS.

Not enough eqns.

Unknowns	Equations
$\bar{u}_i$ – 3	Mass – 1
$\bar{p}$ – 1	Momentum – 1
$\overline{u_i' u_j'}$ – 6	2
8	

Closure problem

~~No~~ ~~#~~ ~~expres~~

Here, constitutive relations cannot be used because there, the stress was fluid property. Here, Reynolds stress is flow property.

2nd order tensor property

$\overline{u_i' u_j'}$ → symmetric.

Let $a_{ij} = \overline{u_i' u_j'} - \frac{1}{3}\overline{u_k' u_k'}\delta_{ij}$

a_{ij} — deviatoric; $\frac{1}{3}\overline{u_k' u_k'}\delta_{ij}$ — isotropic

Turbulent kinetic energy $(k_t) = \frac{1}{2}\overline{u_i' u_i'}$.

$$a_{ij} = \overline{u_i' u_j'} - \frac{2}{3} k_t \delta_{ij}$$

Substituting in eq (B).

$$\rho \frac{\bar{D}}{Dt}(\bar{u_i}) = -\frac{\partial \bar{p}}{\partial x_j} - \frac{\partial}{\partial x_j}(\rho \overline{u_i u_j}) + \frac{\partial}{\partial x_j}\tau_{ij}$$

$$= -\frac{\partial}{\partial x_i}\left(p + \frac{2}{3}\rho k_t\right)\delta_{ij} + \rho \frac{\partial}{\partial x_j}(a_{ij}) + \frac{\partial}{\partial x_j}(\tau_{ij})$$

Energy eqn in Low speed flow

$$\rho C_p \frac{DT}{Dt} = -\frac{\partial \dot{q}_j}{\partial x_j} = k \frac{\partial^2 T}{\partial x_j \partial x_j}$$

(Viscous dissipation neglected because of low speed)

Reynold's average

$$\rho C_p \left[\frac{\partial T}{\partial t} + \frac{\partial (u_j T)}{\partial x_j}\right] = k \frac{\partial^2 T}{\partial x_j \partial x_j} \Rightarrow \rho C_p \frac{\partial \bar{T}}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{u_j}\bar{T} + \overline{u_j' T'}\right) = k \frac{\partial^2 \bar{T}}{\partial x_j \partial x_j}$$

$$\Rightarrow \boxed{\rho C_p \frac{\bar{D}\bar{T}}{Dt} = \frac{\partial}{\partial x_j}\left[k \frac{\partial \bar{T}}{\partial x_j} - \rho C_p \overline{u_j' T'}\right]} \quad \text{——— (1)}$$

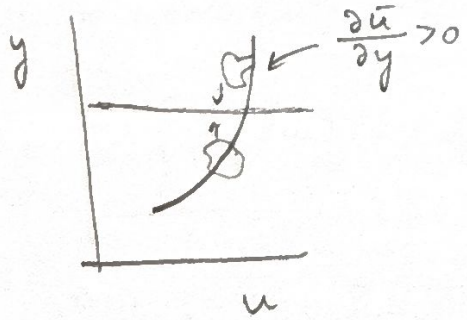
$$\Rightarrow \frac{\bar{D}\bar{T}}{Dt} = \frac{\partial}{\partial x_j}\left[\alpha \frac{\partial \bar{T}}{\partial x_j} - \overline{u_j' T'}\right]$$

$\overline{u_j' T'}$ → Scalar flux ↗ but a vector

Closure problem

$\overline{u_j' T'}$ → Heat transfer due to turbulent fluctuations.

Relation between Reynolds stress and mean velocity gradient.



$y - dy \to y$

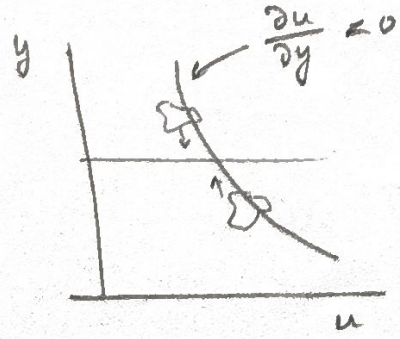
$v' > 0,\ u' < 0$

$\Rightarrow \overline{u'v'} < 0$

$y + dy \to y$

$v' < 0,\ u' > 0$

$\overline{u'v'} < 0$



$y - dy \to y$

$v' > 0,\ u' > 0$

$\overline{u'v'} > 0$

$y + dy \to y$

$v' < 0,\ u' < 0$

$\overline{u'v'} > 0$

Thus, the product becomes,

$$\boxed{-\overline{u'v'}\left(\frac{\partial \bar{u}}{\partial y}\right) > 0}$$ (Always) (Turbulent production term)

Thus, fluid gets energy from turbulence.

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Turbulence modelling

Turbulent viscosity hypothesis (or) Eddy viscosity models (or) Boussinesq method.

Try to connect $-\rho\overline{u_i u_j}$ with velocity (mean).

Deviatoric Reynolds stress $\propto$ mean rate of strain

$$\underbrace{-\rho\overline{u_i u_j} + \frac{2}{3}k_t \rho \delta_{ij}}_{-\rho a_{ij}} = \rho \nu_t 2\bar{S}_{ij}$$

$$= \rho \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

where, $\nu_t \rightarrow$ Turbulent viscosity

depend on flow.

$$\rho \frac{D\bar{u}_i}{Dt} = -\frac{\partial}{\partial x_j}\left[p + \frac{2}{3}k_t \rho\right] + \frac{\partial}{\partial x_j}\left[\rho \nu_{eff}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)\right]$$

where $\nu_{eff} = \nu_{kn} + \nu_t(x,t)$.

ν_{kn} → molecular viscosity

Energy eq

$\overline{u'_i T'} \Rightarrow \overline{u'_i \phi'}$ $\quad \phi \rightarrow$ scalar T.

Gradient - diffusion hypothesis

→ similar to fourier laws of heat conduction

$$\overline{u'\phi'} = -\Gamma_t \nabla\bar{\phi}$$

$$\overline{u'T'} = -\frac{\lambda_t}{\rho c_p}(\nabla \bar{T}).$$

$$\overline{u'T'} = -\alpha_t \Delta\bar{T}$$

$$\frac{D\bar{T}}{Dt} = \frac{\partial}{\partial x_j}\left[\alpha_{eff}\frac{\partial \bar{T}}{\partial x_j}\right]$$

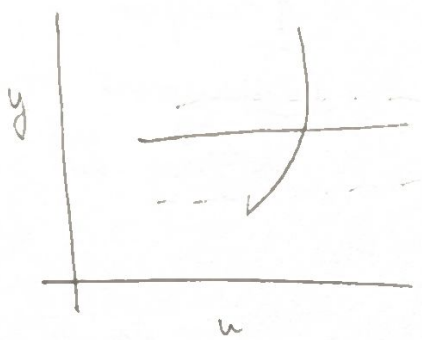
$\lambda_t \rightarrow$ Turbulent conductivity

$\alpha_t \rightarrow$ Turbulent heat diffusivity

$\alpha_{eff} = \alpha + \alpha_t(\underline{x}, t)$.

Prandtl and Mixing Length model

$$\nu_t = u\, l_m$$



$-(\overline{u'v'})\left(\frac{\partial \bar{u}}{\partial y}\right) > 0$

$-\overline{u'v'}$ & $\frac{d\bar{u}}{\partial y}$ are same sign

$\Rightarrow$ ν_t is positive

Take $u' \sim l'\left(\frac{\partial \bar{u}}{\partial y}\right)$

$|v'| \sim |u'|$ (isotropic).

$-\overline{u'v'} \sim l^2\left|\frac{\partial \bar{u}}{\partial y}\right|\left(\frac{\partial \bar{u}}{\partial y}\right)$

l is unknown

$l = (cy)$ where $c \approx 0.4$.

Kinetic energy (Mean) Eq.

$\bar{u}_i \times$ momentum eq. (B)

$$\Rightarrow \bar{u}_i\left\{\frac{\partial(\rho\bar{u}_i)}{\partial t} + \frac{\partial(\rho\bar{u}_i\bar{u}_j)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{\tau}_{ij} - \rho\overline{u_i'u_j'}\right]\right\}$$

$\Rightarrow$ Here, $\cdot\ u_i\frac{\partial(\rho u_i)}{\partial t} = \frac{\partial}{\partial t}\left(\rho\frac{u_i u_i}{2}\right)$. [From (A)]

$\cdot\ u_i\frac{\partial}{\partial x_j}(\rho\bar{u}_i\bar{u}_j) = \frac{\partial}{\partial x_j}\left(\rho\frac{\bar{u}_i\bar{u}_i}{2}\bar{u}_j\right) - \rho\frac{\bar{u}_i\bar{u}_i}{2}\left(\frac{\partial \bar{u}_j}{\partial x_j}\right)^{0}$

$\cdot\ u_i\frac{\partial \bar{P}}{\partial x_i} = \frac{\partial}{\partial x_i}(\bar{u}_i\bar{P}) - \bar{P}\left(\frac{\partial \bar{u}_i}{\partial x_i}\right)^{0}$

$\cdot\ u_i\frac{\partial}{\partial x_j}\left(\bar{\tau}_{ij} - \rho\overline{u_i'u_j'}\right) = \frac{\partial}{\partial x_j}(\bar{u}_i\bar{\tau}_{ij}) - \bar{\tau}_{ij}\frac{\partial u_i}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\rho\bar{u}_i\overline{u_i'u_j'}\right) + \rho\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$

Substituting, Also,

$$\rho \frac{\partial}{\partial t}\left(\frac{\bar{u}_i \bar{u}_i}{2}\right) + \frac{\partial}{\partial x_j}\left(\frac{\rho \bar{u}_i \bar{u}_i \bar{u}_j}{2}\right) = \rho \frac{\bar{D}}{Dt}\left(\frac{\bar{u}_i \bar{u}_i}{2}\right)$$

Substituting,

$$\rho \frac{\bar{D}}{Dt}\left(\frac{\bar{u}_i \bar{u}_i}{2}\right) = \rho \overline{u_i u_j}\left(\frac{\partial \bar{u}_i}{\partial x_j}\right) - \bar{\tau}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial}{\partial x_j}\left[\bar{p}\bar{u}_j - \bar{u}_i \bar{\tau}_{ij} + \bar{u}_i \rho \overline{u_i u_j}\right]$$

$$\rho \frac{\bar{D}}{Dt}\left(\frac{\bar{u}_i \bar{u}_i}{2}\right) = -\bar{P} - \rho \bar{\epsilon} + \nabla \cdot \Pi$$

Transport (How much coming in & going out)

$\bar{P}$ = Production term / shear production

$$= \left(-\rho \overline{u_i u_j}\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)\right) > 0$$

Kinetic energy always decreases.

$\rho \bar{\epsilon}$ = Dissipation mean

Turbulent K.E equation

$$\frac{\bar{D}}{Dt}(\rho k_t) = \bar{P} - \rho \epsilon' - \nabla \cdot \pi'$$

$\bar{P}$ = Production

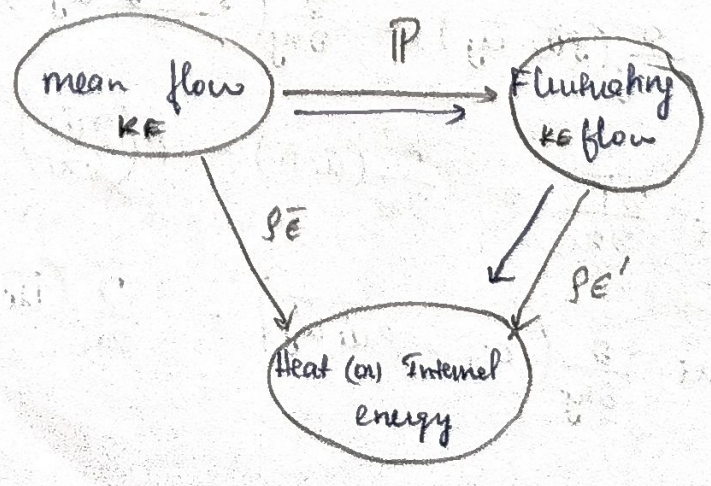
$\rho \bar{\epsilon}'$ = Dissipation due to fluctuation

$$= \overline{\tau'_{ij}\left(\frac{\partial u_i'}{\partial x_j}\right)}$$

$$\pi' = -\tfrac{1}{2}\overline{\rho u_i u_i' u_j'} - \overline{u_j' p'} - \overline{u_i' \tau_{ij}'}$$

Thermal Energy equation

$\rho \bar{\epsilon}$, $\rho \epsilon'$ are +ve



$$\frac{\text{Mean viscous dissipation}}{\text{Shear production}} = \frac{\rho\bar{\epsilon}}{\bar{\mathcal{P}}} = \frac{2\mu\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)}{\rho\,\overline{u_i' u_j'}\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)}$$

Considering $\bar{u}_i \sim U$

$\overline{u_i'} \sim \bar{u}_i \sim U$

$$\tau_{ij}\frac{\partial u_i}{\partial x_j} = \left(\bar{\tau}_{ij}\frac{\partial \bar{u}_i}{\partial x_j}\right) + \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}}$$

$$= \rho\bar{\epsilon} + \rho\bar{\epsilon}'$$

Then $$\frac{\rho\bar{\epsilon}}{\bar{\mathcal{P}}} \sim \frac{2\mu\, U/L}{\rho U^2} \sim \frac{1}{Re_L}$$

For $Re_L >> 1$, $\boxed{\bar{\mathcal{P}} > \rho\bar{\epsilon}}$ ✓

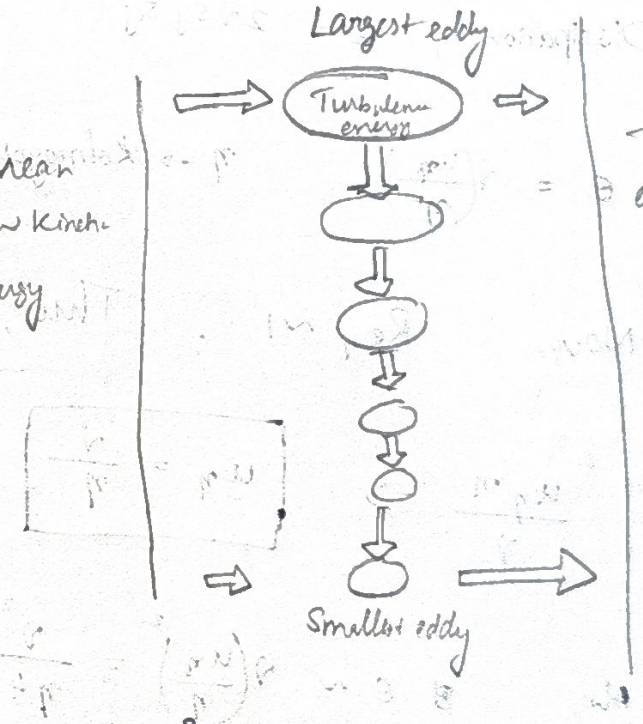
Class.

Energy cascade

Energy → Large scale → Smaller scale → Smaller scale → ⋮ → Smallest scale → Dissipates.

(Mean Kinetic energy)

(Thermal energy)



Eddy → Bulk of fluid moving in coherently.

1) What is the size of smallest eddy in turbulent flow?

2) What are the parameters influencing this?

Kolmogorov Hypothesis.

(i) Kolmogorov Hypothesis of local isotropy

* $Re >> 1$, small scale turbulent motion ($l < l_0$) are statistically isotropic.

$l < l_0$ ← size of large scale.

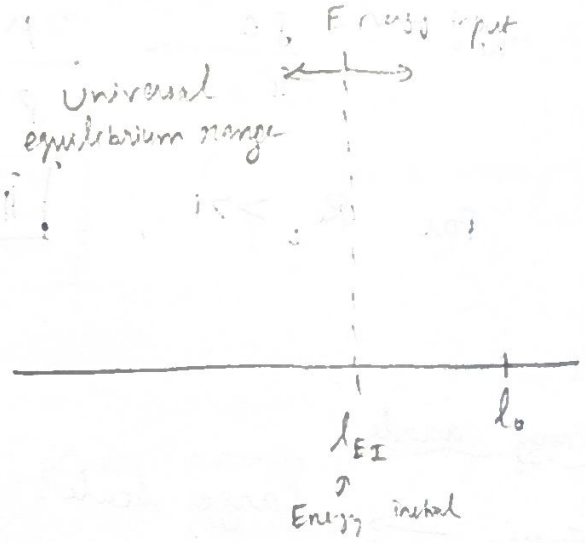
(*) * $Re \gg 1$, small scale turbulent motion is independent of flow configuration (mean flow & BC). $l < l_{EI}$ are isotropic

$l_{EI} \simeq \frac{1}{6} l_0$

(ii) Kolmogorov first similarity hypothesis.

* $Re >> 1$, small scale motion ($l < l_{EI}$) have a universal form which is uniquely determined by ν & ϵ. l

$$l < l_{EI} \Rightarrow \text{universal equilibrium range}$$



Rate of Energy input = Dissipation per mass

$$\left(\tfrac{1}{2}mv^2\right) \quad \dot{w} = \frac{u^2}{t} = \frac{u^3}{l_0}$$

Dissipation, $\epsilon = 2\nu S_{ij} S_{ij}$

$$\epsilon = \nu\left(\frac{u_\eta}{\eta}\right)^2 \qquad \eta \rightarrow \text{Kolmogorov length scale.}$$

Now, $Re_\eta \sim 1$. Thus,

Velocity of smaller eddy $\frac{u_\eta \eta}{\nu} \Rightarrow$ $\boxed{u_\eta = \frac{\nu}{\eta}}$

Also, $\epsilon \sim \nu\left(\frac{u_\eta}{\eta}\right)^2 = \frac{\nu^3}{\eta^4} \Rightarrow \eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$ Length scale.

Thus, $\boxed{u_\eta \sim (\nu\epsilon)^{1/4}}$ $\tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2}$

Ratio between large scale to smallest scale,

$$\frac{l_0}{\eta} \sim Re_L^{3/4}$$

$$\frac{u_0}{u_\eta} \sim Re_L^{1/4}$$

$$\frac{\tau_0}{\tau_\eta} \sim Re_L^{1/2}$$

Example

$L = 0.1\,m$

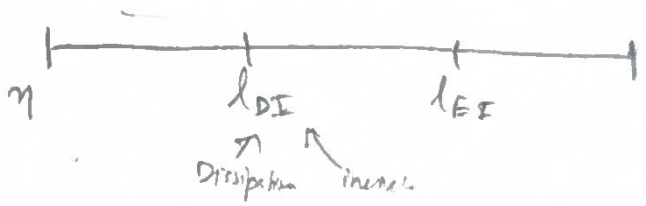
$Re = 10^4$

$\eta \Rightarrow 10^{-4}\,m$

(iii) Kolmogorov second similarity Hypothesis

Pope Turbulence

* $Re \gg 1$, the statistics of the motion of scale l' in the range of $\eta << l << l_0$ have a universal form that is uniquely determined by ϵ, independent of ν.



Turbulent channel flow

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Turbulent Boundary Layer (zero pressure gradient or Flat plate)

Prandtl 3 eqns.

Assumptions

* $Re >> 1$, $\bar{v} << \bar{u}$, $\frac{\partial}{\partial x} << \frac{\partial}{\partial y}$

$\delta/L << 1$

$$\Rightarrow \quad \rho\left[\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right] = -\frac{\partial \bar{P}}{\partial x} + \frac{\partial}{\partial y}\left[\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right]$$

$$0 = -\frac{\partial \bar{P}}{\partial y} + \frac{\partial}{\partial y}\left[-\rho\overline{v'^2}\right]$$

$$\frac{\partial \bar{u}}{\partial x \partial y} + \frac{\partial \bar{v}}{\partial y} = 0$$

Boundary conditions

@ $y = 0$, $\bar{u}, \bar{v} = 0$, $\overline{v'u'} = 0$, $\overline{v'^2} = 0$ (No solution possible)

@ $y = \infty$, $\bar{u} = \bar{u_e}$, $u'v' = 0$

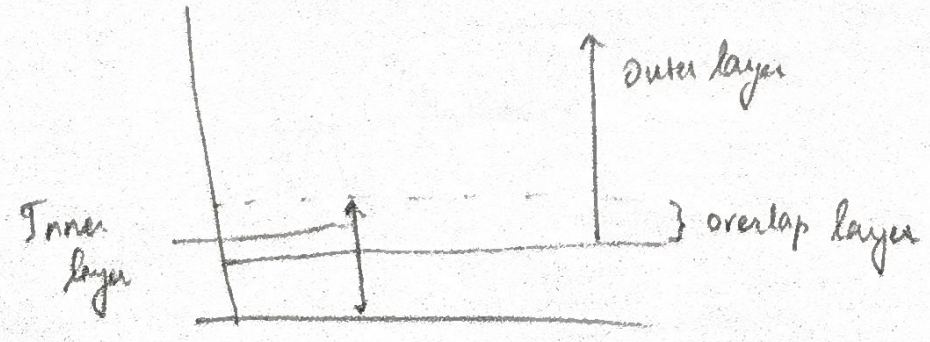
$$\frac{\partial}{\partial y}\left[\bar{P} + \rho\overline{v'^2}\right] = 0$$

$$P + \rho\overline{v'^2} = \text{const} = \bar{P}_{wall} = \bar{P}_e$$

Composite structure of Turbulent BL

To find similarity, we are not able to use a common scale for length & time because of varying viscosity dominance with y.

Thus, divide flow region & apply seperate scaling.



Inner layer
- Overlap layer
- Viscous sublayer
  - Viscous linear sublayer
  - Buffer layer

Outer layer
- Overlap layer
- Wake layer

	Shear stress	Length scale	Velocity scale	Parameter
Wake layer	$-\overline{u'v'}$	δ	$\bar{u}_e$	dP/dx
Overlap layer	$-\overline{u'v'}$	-	$\bar{u}_e$	-
Viscous shear layer	$\mu \frac{\partial \bar{u}}{\partial y}$	l_v	$\bar{u}_e$	K (Surface roughness)

Viscous Sublayer. (smooth wall) $k=0$

$u_\tau \rightarrow$ velocity scale $\quad u_\tau = \sqrt{\frac{\tau_w}{\rho}}$

$l_v \rightarrow$ viscous length scale. $\quad l_v = \frac{\nu}{\bar{u}_\tau}$.

Non-dimensionalize :-

$$u^+ = \frac{\bar{u}}{u_\tau}, \quad y^+ = \frac{y}{l_v} = \frac{\bar{u}_\tau y}{\nu}$$

In this layer $\boxed{u^+ = f(y^+)}$ → "Experimental"

Very near to wall

$$\bar{\tau}_e = \bar{\tau}_w$$

$$\mu \frac{\partial \bar{u}}{\partial y} - \rho \overrightarrow{u'v'}^{\,0} = \rho \bar{u}_\tau^2$$

$$\mu \frac{\partial \bar{u}}{\partial y} = \rho \bar{u}_\tau^2$$

$$\frac{d\bar{u}}{dy} = \frac{\partial \bar{u}}{\partial y} = \frac{\bar{u}_\tau^2}{\nu}$$

Integrating,

$$\bar{u} = \frac{\bar{u}_\tau^2 y}{\nu}$$

$$\boxed{u^+ = y^+}$$ [Viscous linear sublayer]

In Buffer layer, $u^+ = f(y^+)$.

Wake layer

Approximate outerlayer is starting from wall.

Length scale : δ

Velocity scale : $\bar{u}_\tau$ $= u_e$ (Galilean invariance not followed)

Define $\Delta\bar{u} = \bar{u}_e - \bar{u}(y)$

Non-dimensionalize :-

$$\frac{\Delta\bar{u}}{\bar{u}_\tau} = g\left(\frac{y}{\delta}\right)$$

Overlap layer (log layer)

Velocity profile should be continuous. Thus, equate derivative

Viscous layer Now, $\frac{\bar{u}}{\bar{u}_\tau} = f\left(\frac{y}{\nu/\bar{u}_\tau}\right)$ $\left[\because u^+ = f(y^+)\right]$

Differentiate wrt y.

$$\frac{1}{\bar{u}_\tau}\frac{d\bar{u}}{dy} = \frac{df}{dy^+}\frac{dy^+}{dy} = \frac{df}{dy^+}\left(\frac{u_\tau}{\nu}\right)$$

$$\boxed{\frac{d\bar{u}}{dy} = \frac{\bar{u}_\tau^2}{\nu}\frac{df}{dy^+}}$$

From wake layer,

$$\frac{\bar{u}_e - \bar{u}}{\bar{u}_\tau} = g(\eta)$$

Differentiate wrt y

$$\frac{-1}{\bar{u}_\tau}\frac{d\bar{u}}{dy} = \frac{dg}{d\eta}\frac{d\eta}{dy}$$

$$= \frac{dg}{d\eta}\frac{1}{\delta}$$

$$\frac{d\bar{u}}{dy} = -\frac{u_\tau}{\delta}\frac{dg}{d\eta}$$

Equating Slopes,

$$\frac{\bar{u}_\tau^2}{\nu}\frac{df}{dy^+} = -\frac{\bar{u}_\tau}{\delta}\frac{dy}{d\eta}$$

$\times \frac{y}{u_\tau}$

$$\Rightarrow \boxed{y^+\frac{df}{dy^+} = -\eta\frac{dg}{d\eta}}$$

$f(y^+)$ $\quad$ $f(\eta)$

$$y^+\frac{df}{dy^+} = \frac{1}{k}$$

$$-\eta\frac{dg}{d\eta} = \frac{1}{k}$$

Integrating,

$$f = \frac{1}{k}\ln y^+ + B$$

$$\boxed{u^+ = \frac{1}{k}\ln y^+ + B}$$

$$\boxed{\frac{\Delta u}{u_\tau} = -\frac{1}{k}\ln\eta + A}$$

$B \approx 4$ to 5

$A \approx 1$

Velocity varies with log of y. Thus called log layer

Law of wall

* Universal : $u^+ = f(y^+)$ independent of $Re, \frac{dp}{dx}$

class.

Rough wall

* Surface roughness > Viscous layer sub layer.

$k^+ = \frac{k}{l_v}$ $k \rightarrow$ height of surface roughness.

$k^+ < 4 \rightarrow$ hydrodynamically smooth

$k^+ > 70 \rightarrow$ completely rough.

$C_f \simeq$ constant

Zero Pressure Gradient Turbulent BL (Approx.)

$\frac{u}{u_e} = \left(\frac{y}{\delta}\right)^{1/n}$ → Power law.

$n = 6$ to 7.

$\delta^* = \frac{\delta}{n+1}$, $\theta = \frac{n\delta}{(n+1)(n+2)}$. (* Approximations)

For $n = 7$, $\frac{\delta}{x} = \frac{0.37}{Re_x^{1/5}}$

$\frac{\delta^*}{x} = \frac{0.04}{Re_x^{1/5}}$

$\frac{\theta}{x} = \frac{0.036}{Re_x^{1/5}}$

$C_f = \frac{0.057}{Re_x^{1/5}}$

Thus

Laminar $\propto \frac{1}{Re^{0.5}}$

Turbulent $\propto \frac{1}{Re^{0.2}}$

	Friction drag	Heat transfer	Flow separation
Laminar	Low	low	less APG
Turbulent	High	high	high APG

TRANSITION

Transition (process happens over a length)

├ First step — Flow instability

1) Natural transition → Tollemin- Schlichting wave (TS).

2) Bypass

3) Forced

* Natural transition — TS waves

Spanwise vorticity

Turbulent flow

* Bypass → Directly to turbulent flow

→ Due to external noise & vibration.

* Forced → Surface imperfection

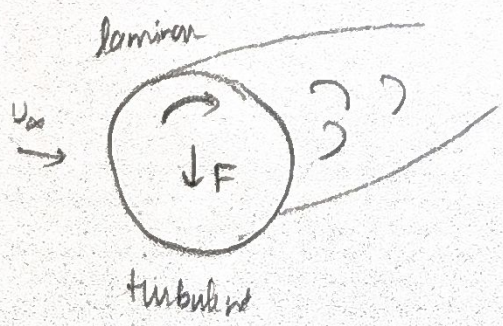
Flow seperation

what happens to seperation point?

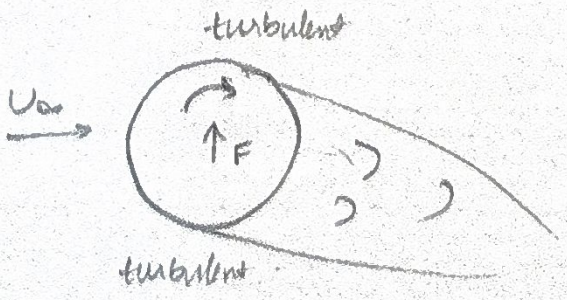
Boundary layer eqns are not valid for flow seperation.

wherever pressure remains approx. constant, flow seperation ~~kept~~ may happen.

Flow seperation modifies the pressure distribution ⇒ changes drag & lift.



$Re < Re_{cr}$



$Re > Re_{cr}$